SINGULAR REDUCTION AND QUANTIZATION

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Abstract. Consider a compact prequantizable symplectic manifold M on which a compact Lie group G acts in a Hamiltonian fashion. The "quantization commutes with reduction" theorem asserts that the G-invariant part of the equivariant index of M is equal to the Riemann-Roch number of the symplectic quotient of M, provided the quotient is nonsingular. We extend this result to singular symplectic quotients, using partial desingularizations of the symplectic quotient to define its Riemann-Roch number. By similar methods we also compute multiplicities for the equivariant index of the dual of a prequantum bundle, and furthermore show that the arithmetic genus of a Hamiltonian G-manifold is invariant under symplectic reduction.

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1. Introduction

Consider a compact symplectic manifold (M, ω) on which a compact Lie group G acts in a Hamiltonian fashion with equivariant moment map $\Phi\colon M \to \mathfrak{g}^*$. A fundamental result due to Meyer and Marsden-Weinstein says that if 0 is a regular value of Φ, then the *symplectic quotient* (also known as the *reduced space*)

$$M_0 = \Phi^{-1}(0)/G$$

is a symplectic orbifold, a symplectic space with finite-quotient singularities. However, if 0 is a singular value of the moment map, the symplectic quotient usually has more complicated singularities. Singular symplectic quotients have been the subject of intensive study over the past fifteen years. For instance, it was proved by Arms et al. [1] and Otto [37] that M_0 admits a finite decomposition into smooth symplectic manifolds, labelled by orbit types of M. Sjamaar and Lerman [41] proved that this decomposition is a stratification in the sense of Thom-Mather and gave local normal forms for the singularities.

The object of this paper is twofold. The first goal is to define Riemann-Roch numbers of singular symplectic quotients M_0 with coefficients in certain complex line bundles. These bundles include the trivial line bundle, the Riemann-Roch number of which we call the arithmetic genus of M_0, and the prequantum line "bundle", the Riemann-Roch number of which is the dimension of the quantization of M_0. (The prequantum line "bundle" is not a genuine fibre bundle but an orbibundle.)

The second goal is to understand how these Riemann-Roch numbers are related to the corresponding characteristic numbers of M. Our incentive is to extend the "quantization commutes with reduction" theorems of [34, 22] to the singular case. These theorems arose from a conjecture of Guillemin and Sternberg [18].

A major obstacle to attaining our first goal is the fact that on a singular space there is no obvious way to define a Riemann-Roch number as the index of an elliptic

operator. To make matters worse, symplectic quotients are seldom complex algebraic or even analytic varieties, so that holomorphic Euler characteristics do not make sense and the Riemann-Roch formulas of Baum et al. [4] and Levy [30] do not apply. Our attempt to surmount this obstacle consists in (partially) resolving the singularities of M_0 and defining the Riemann-Roch numbers of M_0 to be equal to those of its desingularization. This raises the question whether the result depends on the way in which we resolve the singularities. In contrast to the situation in algebraic geometry this is not an easy question, and the answer we find is incomplete. One way of desingularizing a symplectic quotient was discovered by Kirwan [25]. Another way is simply to shift the value of the moment map to a nearby generic value. Our result says that these two desingularization methods lead to the same Riemann-Roch numbers. (Neither method yields a desingularization of M_0 in the strict sense of the word, but only a partial desingularization, which may have finite-quotient singularities.)

We are far more successful in winning our second objective. Let L be a G-equivariant complex line bundle on M. Let $\mathrm{RR}(M, L)$ be the equivariant index of M with coefficients in L, that is the pushforward of L to a point, viewed as an element of the equivariant K-theory of a point. Under favourable circumstances, e. g. if L is the trivial bundle or the prequantum bundle, L induces a line "bundle" L_0 on the quotient M_0. This enables us to define the Riemann-Roch number of M_0 with coefficients in L_0 by means of either of the two desingularization processes referred to above. Our results include:

1. if L is trivial, then $\mathrm{RR}(M, L) = \mathrm{RR}(M_0, L_0)$, i. e. the arithmetic genus of M is equal to the arithmetic genus of M_0;

2. if L is the prequantum line bundle, then $\mathrm{RR}(M, L)^G = \mathrm{RR}(M_0, L_0)$, i. e. quantization commutes with reduction. (The superscript G denotes G-invariants.)

The latter result leads to a geometric formula for the multiplicities of all irreducible representations occurring in the quantization $\mathrm{RR}(M, L)$. We obtain similar results for the "negative" quantization $\mathrm{RR}(M, L^{-1})$.

It also turns out that the multiplicities depend only on the weights of the action of T on the fibres of L at the fixed-point set M^T, where T denotes a maximal torus of G. This observation enables us to generalize our results to a larger class of bundles.

The method of proof is an extension to the singular case of techniques developed in [34] for the proof of the Guillemin-Sternberg conjecture, the key tool being a gluing formula that relates the equivariant index of M to equivariant indices of simpler spaces obtained from M by symplectic cutting in the sense of Lerman [27]. The gluing formula is an application of the Atiyah-Segal-Singer equivariant index formula.

Because the operations of symplectic cutting and partial desingularization give rise to orbifolds rather than manifolds and because many "bundles" we shall consider are orbibundles rather than bundles, we are obliged to place our discussion within the wider framework of Hamiltonian G-orbifolds. While this presents few conceptual difficulties, the technicalities are sometimes rather involved. In the interests of clarity and brevity we shall at some points treat in complete detail only the manifold case and indicate succinctly how to extend the argument to orbifolds. The relevant versions of the index formula in this category are due to Kawasaki [23], Vergne [44] and Duistermaat [12].

The organization of this paper is as follows. Section 2 contains detailed statements and a discussion of our main results. In Section 3 we review the local structure of singular symplectic quotients. In Section 4 we describe the two known methods for desingularizing symplectic quotients. We prove in detail that Kirwan's partial desingularization is well-defined up to deformation equivalence and discuss briefly how it is related to shift-desingularizations. We then present the proofs of our main results, first in the abelian case (Section 5), then in the nonabelian case (Section 6). At several points we illustrate our results by applying them to Delzant spaces, a class of toric varieties with symplectic structures. These not only serve as an interesting example, but also play an important part in symplectic cutting. Appendix A contains a number of technical results concerning blowups and constant-rank embeddings. In Appendix B we prove a product formula for the Todd class of an almost complex fibre bundle, which generalizes a classical result of Borel. A table listing our notational conventions is provided in Appendix C.

2. Statement of results

In Section 2.1 we introduce notation and some basic notions concerning Hamiltonian actions. This is standard material except for Definitions 2.1 and 2.2. Sections 2.2–2.5 are a compendium of the chief results of this paper. It is important to note that these results hold without any regularity assumptions on the values of the moment map.

2.1. **Preliminaries.** Throughout this paper G denotes a compact connected Lie group. We choose once and for all a maximal torus T of G and a (closed) Weyl chamber $\mathfrak{t}^*_+$ in $\mathfrak{t}^*$ and denote by $\mathfrak{W}$ the Weyl group $N_G(T)/T$. Throughout (M, ω, Φ) designates a connected symplectic orbifold on which G acts in a Hamiltonian fashion with a proper and G-equivariant moment map $\Phi\colon M \to \mathfrak{g}^*$. (Many of our results hold only for compact M, but it is important for technical reasons to allow M to be noncompact.) Our sign convention for the moment map is as follows:

$$\mathrm{d}\langle \Phi, \xi \rangle = \iota(\xi_M)\omega,$$

where ξ_M denotes the fundamental vector field induced by $\xi \in \mathfrak{g}$. The pair (ω, Φ) is an *equivariant symplectic form* on M. An *isomorphism* between two Hamiltonian G-orbifolds is a G-equivariant symplectomorphism that intertwines the moment maps on the two spaces. Some basic material on symplectic orbifolds can be found in [28, 29, 34]. Our conventions concerning orbifolds are as in [34]. (In particular, the structure group of an orbifold at a point is not required to act effectively, and the structure group of a suborbifold at a point is the same as the structure group of the ambient orbifold at that point.) The set $\Phi(M) \cap \mathfrak{t}^*_+$ will be denoted by Δ. By a theorem of Kirwan [24] (cf. also [28, 38]) it is a convex rational polyhedron, referred to as the *moment polyhedron*. For $\mu \in \mathfrak{g}^*$ let G_μ be the stabilizer group of μ under the coadjoint action and let $G\mu$ be the coadjoint orbit through μ. The compact space

$$M_\mu = \Phi^{-1}(\mu)/G_\mu \cong \Phi^{-1}(G\mu)/G$$

is the *symplectic quotient* of M at level μ. The symplectic quotient at level 0 plays a special role and (particularly in situations where there is more than one group acting) will also be denoted by

$$M/\!\!/G = M_0.$$

The symplectic quotients of M have a natural stratification by symplectic orbifolds determined by the infinitesimal orbit types of M. (See Section 3.1.)

Definition 2.1. A point $\mu \in \mathfrak{g}^*$ is a *quasi-regular* value of Φ if the G-orbits in $\Phi^{-1}(G\mu)$ all have the same dimension.

Equivalently, μ is quasi-regular if the rank of Φ is constant on $\Phi^{-1}(G\mu)$, or the dimension of the stabilizer G_m is the same for all m in $\Phi^{-1}(G\mu)$, that is to say if $\Phi^{-1}(G\mu)$ is contained in a single infinitesimal orbit type stratum. Consequently, if μ is a quasi-regular value, then the orbifold stratification of M_μ consists of one piece only, and M_μ is therefore a symplectic orbifold. Here are some examples of quasi-regular values: weakly regular values (i. e. values μ for which Φ intersects $\{\mu\}$ cleanly; see Proposition 3.10), points in Δ of maximal norm (see Lemma 6.1) and, if G is abelian, vertices of the moment polytope. See Section 3.4 for more examples.

Let L be a G-equivariant complex line orbibundle (also known as an orbifold line bundle) on M. For $\mu \in \mathfrak{g}^*$ define L_μ to be the quotient of the restriction of L to $\Phi^{-1}(G\mu)$,

$$L_\mu = \left(L|_{\Phi^{-1}(G\mu)}\right)/G.$$

For $\mu = 0$ we shall also use the notation

$$L/\!\!/G = L_0.$$

Suppose for a moment that L is a true line bundle. When is L_μ a topologically locally trivial complex line bundle on the topological space M_μ? It is not hard to see that this is the case if and only if for all $m \in \Phi^{-1}(G\mu)$ the stabilizer G_m acts trivially on the fibre L_m, in other words $L|_{\Phi^{-1}(G\mu)}$ is G-equivariantly locally trivial. We shall make a slightly weaker assumption.

Definition 2.2. The line orbibundle L is *almost equivariantly locally trivial* at m if the action of the identity component of G_m on L_m is trivial. It is *almost equivariantly locally trivial at level* μ if it is almost equivariantly trivial at all $m \in \Phi^{-1}(G\mu)$.

For instance, if μ is a regular value of Φ, then G acts locally freely on $\Phi^{-1}(G\mu)$, so L is almost equivariantly locally trivial at μ. If L is almost equivariantly locally trivial at μ, then the fibres of the induced map $L_\mu \to M_\mu$ are finite quotients of $\mathbb{C}$. If in addition μ is a quasi-regular value, then L_μ is a line orbibundle over the orbifold M_μ.

Now assume that M is compact. Choose a G-invariant almost complex structure J on M which is compatible with ω in the sense that the symmetric bilinear form $\omega(\cdot, J\cdot)$ is a Riemannian metric. Let $\bar\partial_L$ be the Dolbeault operator with coefficients in L. Also choose a G-invariant Hermitian fibre metric on L. The *Dolbeault-Dirac operator* on M with coefficients in L is defined by $\not{\partial}_L = \sqrt{2}\,(\bar\partial_L + \bar\partial_L^*)$, considered as an operator from $\Omega^{0,\mathrm{even}}(M, L)$ to $\Omega^{0,\mathrm{odd}}(M, L)$. The *equivariant Riemann-Roch number* of M with coefficients in L is the equivariant index of $\not{\partial}_L$,

$$\mathrm{RR}(M, L) = \mathrm{index}_G(\not{\partial}_L),$$

viewed as an element of $\mathrm{Rep}\,G$, the character ring of G. An alternative definition goes as follows. Every G-orbifold with an invariant almost complex structure carries a canonical invariant Spin_c-structure. The Spin_c-Dirac operator of M with

coefficients in L has the same principal symbol as $\partial\!\!\!/_L$ (see e. g. [12]), and therefore has the same equivariant index.

The character $\mathrm{RR}(M, L)$ does not depend on the choice of J (because any two compatible almost complex structures are homotopic), nor on the choice of the fibre metric on L. Indeed, $\mathrm{RR}(M, L)$ depends only on the homotopy class of the almost complex structure and the equivariant Chern class of L.

Let Λ be the integral lattice $\ker\bigl(\exp|_{\mathfrak{t}}\bigr)$ of $\mathfrak{t}$. Then $\Lambda^* = \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$ is the lattice of real infinitesimal weights and $\Lambda^*_+ = \Lambda^* \cap \mathfrak{t}^*_+$ is the set of real dominant weights. The *multiplicity function* of L is the function N_L (also denoted by N) on Λ^*_+ with values in $\mathbb{Z}$ defined by the orthogonal decomposition

$$\mathrm{RR}(M, L) = \sum_{\mu \in \Lambda^*_+} N_L(\mu)\, \chi_\mu, \tag{2.1}$$

where χ_μ denotes the character of the irreducible representation with highest weight μ.

Even if M is not compact, its symplectic quotients are, so if μ is a quasi-regular value of Φ and L is almost equivariantly locally trivial at μ, then the Riemann-Roch number $\mathrm{RR}(M_\mu, L_\mu)$ of the orbifold M_μ with coefficients in the orbibundle L_μ is well-defined. We shall now discuss how to define $\mathrm{RR}(M_\mu, L_\mu)$ even when M_μ is not an orbifold.

2.2. **The singular case.** Consider a value μ of the moment map that is not quasi-regular. Let L be a G-equivariant line orbibundle on M and suppose that L is almost equivariantly locally trivial at μ. Every point in M_μ has an open neighbourhood O which can be written as a quotient of a space $\tilde{O}$ by a finite group Γ such that L_μ is the quotient by Γ of a Γ-equivariant line bundle on $\tilde{O}$. (See Section 3.3.) We shall call L_μ a line orbibundle over M_μ, even though the base space M_μ need not be an orbifold. Now let $\tilde{M}_\mu$ be Kirwan's canonical partial desingularization of M_μ. To construct $\tilde{M}_\mu$ one first performs a sequence of equivariant symplectic blowups of a G-invariant neighbourhood U of $\Phi^{-1}(G\mu)$ to obtain a Hamiltonian G-orbifold $(\tilde{U}, \tilde{\omega}, \tilde{\Phi})$ with the property that μ is a quasi-regular value of $\tilde{\Phi}$. The space $\tilde{M}_\mu$ is then the reduction of $\tilde{U}$ at μ, which is a symplectic orbifold. The pullback bundle $\tilde{L}$ on $\tilde{U}$ is almost equivariantly locally trivial at μ, so that $\tilde{L}_\mu$ is a line orbibundle over $\tilde{M}_\mu$ and the Riemann-Roch number $\mathrm{RR}(\tilde{M}_\mu, \tilde{L}_\mu)$ makes sense.

Definition 2.3. $\mathrm{RR}(M_\mu, L_\mu) = \mathrm{RR}\bigl(\tilde{M}_\mu, \tilde{L}_\mu\bigr)$.

In the algebraic case, where M is a complex projective orbifold and the quotients M_μ are complex projective varieties, this equality is not the definition of $\mathrm{RR}(M_\mu, L_\mu)$, but a consequence of the fact that the M_μ have rational singularities. (Cf. [39].)

One problem with this definition is that the symplectic structures on $\tilde{U}$ and $\tilde{M}_\mu$ depend on a long list of choices. However, in Section 4.2 we prove the following result.

Theorem 2.4. *The germ at $\tilde{\Phi}^{-1}(G\mu)$ of the triple $\bigl(\tilde{U}, \tilde{\omega}, \tilde{\Phi}\bigr)$ is unique up to deformation equivalence. This implies that the symplectic structure on $\tilde{M}_\mu$ is unique up to deformation equivalence and hence that the Riemann-Roch numbers of $\tilde{M}_\mu$ are well-defined.*

A more difficult and as yet unresolved problem is that the partial resolution $\tilde{M}_\mu$ depends on the way M_μ is written as a quotient. It is conceivable that M_μ could be presented in a different way as a quotient $M'_{\mu'}$ of a Hamiltonian G'-orbifold M' and it is *a priori* unclear if $\tilde{M}_\mu$ and $\tilde{M}'_{\mu'}$ have the same Riemann-Roch numbers. The theorem below offers limited evidence that $\mathrm{RR}(M_\mu, L_\mu)$ is independent of the partial resolution. By Lemma 3.7 almost equivariant local triviality is an open condition, so for all ν near μ the quotient L_ν is a line orbibundle over M_ν and $\mathrm{RR}(M_\nu, L_\nu)$ is well-defined (as $\mathrm{RR}(\tilde{M}_\nu, \tilde{L}_\nu)$ if M_ν is singular).

Theorem 2.5. *If L is almost equivariantly locally trivial at level μ, then*

$$\mathrm{RR}(M_\mu, L_\mu) = \mathrm{RR}(M_\nu, L_\nu)$$

for all $\nu \in \Phi(M)$ sufficiently close to μ.

The proof is in Section 6.1. If M_μ and M_ν have the same dimension (e. g. if μ and ν are both in the interior of Δ), then $\tilde{M}_\mu$ and $\tilde{M}_\nu$ are two different partial desingularizations of M_μ, and we shall refer to $\tilde{M}_\nu$ as a *shift desingularization* of M_μ. Theorem 2.5 asserts that the shift desingularizations of M_μ give the same Riemann-Roch numbers as the canonical partial desingularization.

In the next sections we shall make a detailed comparison between the virtual character $\mathrm{RR}(M, L)$ and the numbers $\mathrm{RR}(M_\mu, L_\mu)$ for three different types of bundle.

2.3. **Rigid bundles.** A G-equivariant line orbibundle L on M is called *rigid* (or *G-rigid*) if the action of T on $L|_{M^T}$ is trivial. This condition is obviously independent of the choice of the maximal torus T. A rigid bundle is almost equivariantly locally trivial everywhere by Lemma 3.11, so that $\mathrm{RR}(M_\mu, L_\mu)$ is well-defined for all μ.

Example 2.6. The equivariantly trivial line bundle $\mathbb{C}$ is rigid. Its equivariant index is the *arithmetic* or *Todd genus* of M. The induced bundle $\mathbb{C}_\mu$ is of course the trivial line bundle on M_μ.

Notice that the definition of a rigid orbibundle makes sense for an arbitrary almost complex G-orbifold. The term “rigid orbibundle” is inspired by an observation of Lusztig (see [2]) stating that the arithmetic genus of a almost complex G-manifold is rigid (i. e. a constant character). Along the lines of [2, 6, 13, 26] we shall prove the following stronger result.

Theorem 2.7.

1. *Let $\mathcal{M}$ be a compact almost complex G-orbifold and let $\mathcal{L}$ be a rigid orbibundle on $\mathcal{M}$. Then the character $\mathrm{RR}(\mathcal{M}, \mathcal{L})$ is constant. Choose a generic $\xi \in \mathfrak{t}$. Then*

$$\mathrm{RR}(\mathcal{M}, \mathcal{L}) = \sum_{\substack{F \\ \xi \in \check{\mathcal{C}}_F}} \mathrm{RR}(F, \mathcal{L}|_F),$$

where the summation is over all connected components F of $\mathcal{M}^T$ such that $\langle \alpha, \xi \rangle < 0$ for all orbiweights α of the T-action on the normal bundle of F.

2. *Let M be a compact Hamiltonian G-orbifold and let L be a rigid orbibundle on M. Then*

$$\mathrm{RR}(M, L) = \mathrm{RR}(M_\mu, L_\mu) \tag{2.2}$$

for all $\mu \in \Phi(M)$. In particular, $\mathrm{RR}(M_\mu, L_\mu)$ does not depend on the value of μ. In the presence of an action of a compact connected Lie group H on

M and L that commutes with the action of G, (2.2) *holds as an equality of virtual characters of H.*

The proof of 1 is in Section 5.1 and the proof of 2 is in Section 6.1. Setting $L = \mathbb{C}$ we obtain from 2 that the arithmetic genus is invariant under symplectic reduction and that all symplectic quotients of M have the same arithmetic genus. The latter fact is perhaps not very surprising (and was known in special cases; see e. g. [3]), since according to Guillemin and Sternberg [19] all nonsingular symplectic quotients are "birationally equivalent" in the symplectic category. For regular values μ the invariance of the arithmetic genus under reduction was proved independently by Tian and Zhang [42].

Applying this result to a coadjoint orbit $M = G\mu$, we find that the arithmetic genus of M is equal to 1 since the symplectic quotient M_μ is a point. (This observation follows also from Theorem 2.23 in [39].) This implies that any two-dimensional symplectic quotient of a coadjoint orbit is a sphere (possibly with orbifold singularities).

2.4. **Moment bundles.** A G-equivariant line orbibundle L over M is called a *moment bundle* (or *G-moment bundle*) if for all components F of the fixed-point set M^T the orbiweight of the T-action on $L|_F$ is equal to $\iota^*\Phi(F)$. Here ι denotes the inclusion map $\mathfrak{t} \to \mathfrak{g}$ and $\iota^*\Phi(F)$ the (constant) value of $\iota^*\Phi$ on F. It is easy to see that this condition is independent of the choice of the maximal torus T. It is also obvious that if M admits a moment bundle L, then its moment polytope is rational. In fact, $\iota^*\Phi(F) \in d^{-1}\Lambda^*$ if the generic fibre of $\mathcal{L}|_F$ is the folded line $\mathbb{C}/(\mathbb{Z}/d\mathbb{Z})$. If M is compact, a moment bundle is almost equivariantly locally trivial at 0 by Lemma 3.11, so that $\mathrm{RR}(M_0, L_0)$ is well-defined.

Example 2.8. A *prequantum line bundle* is a G-equivariant line orbibundle whose equivariant Chern class is equal to the equivariant cohomology class of the equivariant symplectic form (ω, Φ). The equivariant index of a prequantum line bundle is called the *quantization* of M. A prequantum line bundle always exists if the cohomology class of ω is integral and G is simply connected and is then a true G-equivariant line bundle. (A nessesary and sufficient condition is the integrality of the G-equivariant *orbifold* cohomology class of ω.) On a prequantum line bundle L there exist a G-invariant Hermitian metric and connection such that the curvature of the connection is equal to the symplectic form and for every $\xi \in \mathfrak{g}$ the fundamental vector field ξ_L is given by Kostant's formula

$$\xi_L = \operatorname{lift} \xi_M + \langle \Phi, \xi \rangle \frac{\partial}{\partial \phi}. \tag{2.3}$$

Here $\partial/\partial\phi$ is the generating vector field for the scalar S^1-action on L. This implies immediately that L is a moment bundle. If 0 is a quasi-regular value, then the reduced bundle L_0 is a prequantum line bundle on the symplectic orbifold M_0. (See e. g. [18].) If 0 is not quasi-regular we shall still call the orbibundle L_0 a prequantum line bundle on the stratified symplectic space M_0. Note however that for $\mu \neq 0$ the quotient bundles L_μ are *not* prequantizing.

Our main result is as follows. The proof is in Section 6.2.

Theorem 2.9. *Let M be a compact Hamiltonian G-orbifold and L be a G-moment bundle on M. Then the multiplicity of the trivial representation in* $\mathrm{RR}(M, L)$ *is*

equal to the Riemann-Roch number of the symplectic quotient M_0*:*

$$\mathrm{RR}(M,L)^G = \mathrm{RR}(M_0,L_0). \tag{2.4}$$

In the presence of an action of a compact connected Lie group H *on* M *and* L *that commutes with the action of* G*,* (2.4) *holds as an equality of virtual characters of* H*. By Theorem 2.5 we also have* $\mathrm{RR}(M,L)^G = \mathrm{RR}(M_\mu,L_\mu)$ *for small* $\mu \in \Phi(M)$*.*

For prequantum line bundles this result goes by the name of "quantization commutes with reduction" and was conjectured (for regular values of the moment map) by Guillemin and Sternberg [18]. Results on the quantization conjecture were obtained in [18], [39] and [8] in the context of Kähler quantization. In the above formulation the quantization conjecture was first proved (for regular values) by Guillemin [15], Meinrenken [35] and Vergne [43] in the abelian case and by Meinrenken [34] in the nonabelian case. A similar result was obtained by Jeffrey and Kirwan [22]. For a presymplectic version see Canas et al. [10]. A proof of the quantization conjecture using analytical methods was given (in the regular case) by Tian and Zhang [42]. See [36] for an application to loop group actions. See [40] for a survey and further references.

Combined with the shifting trick Theorem 2.9 leads to a complete decomposition of the virtual character $\mathrm{RR}(M,L)$ into irreducible characters as follows. Consider a moment bundle L on M. Identify $\mathfrak{t}^*$ with the subspace of T-fixed vectors in $\mathfrak{g}^*$. Recall that every weight μ exponentiates to a character $G_\mu \to S^1$, which gives rise to a line bundle

$$E_\mu = G \times^{G_\mu} \mathbb{C}$$

on the symplectic manifold $G\mu$. The unique compatible invariant almost complex structure on $G\mu$ is integrable, and by the Borel-Weil-Bott theorem the equivariant Riemann-Roch number satisfies

$$\mathrm{RR}(G\mu, E_\mu) = \mathrm{Ind}_T^G \zeta_\mu. \tag{2.5}$$

Here ζ_μ is the character of T defined by

$$\zeta_\mu(\exp\xi) = \exp 2\pi i\mu(\xi)$$

and $\mathrm{Ind}_T^G\colon \mathrm{Rep}\,T \to \mathrm{Rep}\,G$ denotes the induction functor, which is defined as follows. Let $w \odot \mu = w(\mu+\rho) - \rho$ denote the affine action of the Weyl group $\mathfrak{W}$, where ρ is half the sum of the positive roots. If $\mu \in \Lambda^*$ then $\mathrm{Ind}_T^G \zeta_\mu$ is nonzero if and only if there exists a Weyl group element w with $w \odot \mu \in \mathfrak{t}^*_+$ and in this case

$$\mathrm{Ind}_T^G \zeta_\mu = (-1)^{\mathrm{length}(w)} \chi_{w\odot\mu}. \tag{2.6}$$

Let $*$ denote the involution of $\mathfrak{t}$ defined by $*\mu = \mu^* = -w_0\mu$, where w_0 is the longest Weyl group element. Then for dominant μ we have by (2.5) and the Künneth formula

$$N(\mu) = \mathrm{RR}(M \times G\mu^*, L \boxtimes E_{\mu^*})^G,$$

where $N = N_L$ is the multiplicity function of L. It is easy to see that $L \boxtimes E_{\mu^*}$ is a moment bundle on the product $M \times G\mu^*$, so we can use Theorem 2.9 to evaluate the right-hand side. According to the shifting trick, $(M \times G\mu^*)/\!\!/G \cong M_\mu$.

Definition 2.10. The *shifted quotient bundle* on M_μ is the orbibundle $L_\mu^{\mathrm{shift}} = (L \boxtimes E_{\mu^*})/\!\!/G$.

Observe that L_μ^{shift} is not equal to L_μ unless $\mu = 0$. In fact, the shifted quotient bundle cannot even be defined unless μ is integral! If L is a prequantum line bundle, then L_μ^{shift} is a prequantum line bundle on M_μ. Theorem 2.9 implies $N(\mu) = \mathrm{RR}(M_\mu, L_\mu^{\mathrm{shift}})$, so we have proved the following statement.

Corollary 2.11. *Let L be a moment bundle on the compact Hamiltonian G-orbifold M. Then the decomposition of* $\mathrm{RR}(M, L)$ *into irreducible characters is given by*

$$\mathrm{RR}(M, L) = \sum_{\mu \in \Lambda^* \cap \Delta} \mathrm{RR}(M_\mu, L_\mu^{\mathrm{shift}})\, \chi_\mu.$$

In particular, the support of the multiplicity function is contained in the moment polytope Δ. □

While this formula may be difficult to evaluate in practice (unless the quotients are zero- or two-dimensional), in combination with the index theorem for orbifolds it yields interesting qualitative information about the multiplicity diagram of a moment bundle L. For instance, a weaker form of Corollary 2.11 was used in [35] to prove a quantum version of the Duistermaat-Heckman Theorem. We shall now use Corollary 2.11 to improve on this result.

Let M_{prin} be the principal infinitesimal orbit type stratum, i. e. the set of all points at which the stabilizer has minimal dimension. Then Φ has maximal rank on M_{prin} and the quotients M_μ for $\mu \in \Phi(M_{\mathrm{prin}})$ all have the same dimension, say $2k$. Let $\operatorname{int}\Delta$ be the relative interior of the polytope Δ. The set of *generic values* of Φ is the set Δ_{gen} consisting of all $\mu \in \operatorname{int}\Delta$ satisfying $\Phi^{-1}(\mu) \subset M_{\mathrm{prin}}$. Let

$$\Delta_{\mathrm{gen}} = \bigcup_i \Delta_i \tag{2.7}$$

be its decomposition into connected components. The closure of each component Δ_i is a convex polytope (see e. g. [28]), and $\Phi\colon \Phi^{-1}(\Delta_i) \to \Delta_i$ is a locally trivial fibre bundle. The *Duistermaat-Heckman measure* is the measure λ_{DH} on $\mathfrak{t}^*$ defined by

$$\lambda_{\mathrm{DH}}(\mu) = \mathrm{vol}(M_\mu)\, \lambda(\mu),$$

where $\mathrm{vol}(M_\mu)$ denotes the $2k$-dimensional symplectic volume of M_μ and λ the normalized Lebesgue measure on the affine subspace spanned by Δ. According to the Duistermaat-Heckman Theorem the density function $\mu \mapsto \mathrm{vol}(M_\mu)$ is continuous on Δ and is given by a polynomial on Δ_i for every i.

Recall that a function $f\colon \Xi \to \mathbb{Z}$ defined on a lattice $\Xi \cong \mathbb{Z}^r$ is called *quasi-polynomial* if there exists a sublattice Ξ' of finite index such that for all $\gamma \in \Xi$ the translates $f_\gamma = f(\gamma + \cdot)\colon \Xi' \to \mathbb{Z}$ are polynomial functions. The degree of the f_γ is called the *degree* of f. If Ξ' is chosen as large as possible, the number of elements in Ξ/Ξ' is the *period* of f. Consider for instance the lattice $\Xi = \mathbb{Z} \times \Lambda^*$ and the function

$$f(m, \mu) = N^{(m)}(\mu),$$

where $N^{(m)}(\mu)$ is defined as the multiplicity of μ in the character $\mathrm{RR}(M, L^m)$. Replacing ω by $m\omega$ and Φ by $m\Phi$ we obtain from Corollary 2.11 expressions for $N^{(m)}(\mu)$ for all $m > 0$ and from Theorem 2.7 for $m = 0$. (This does not work for $m < 0$ because then the almost complex structure on M is not tame with respect to $m\omega$.) From Kawasaki's Riemann-Roch formula applied to the orbifolds M_μ (or $\tilde{M}_\mu$ for μ that are not quasi-regular) we then read off the following result.

Corollary 2.12 (quantum DH). *For every moment bundle L on M the function $(m,\mu) \mapsto N^{(m)}(\mu)$ is quasi-polynomial on each of the closed cones*

$$\mathcal{C}_i = \{\, (t, t\mu) : t \geq 0 \text{ and } \mu \in \bar{\Delta}_i \,\}.$$

Each of these quasi-polynomials has degree $\leq k$, where $2k$ is the dimension of the generic symplectic quotient, and the degree is equal to k for all i if L is a prequantum bundle. For $m \geq 0$ the function $m \mapsto N^{(m)}(0)$ is a quasi-polynomial, whose period is a divisor of the smallest positive integer l such that the quotient bundle $L^l /\!\!/ G$ is a genuine line bundle, i. e. has fibre equal to $\mathbb{C}$ everywhere. □

The fact that the multiplicities $N^{(m)}(\mu)$ exhibit quasi-polynomial behaviour even on the boundary of the cones $\mathcal{C}_i$ may be viewed as a quantum version of the continuity property of the Duistermaat-Heckman measure at the walls of Δ.

2.5. **Dual moment bundles.** Let L be a moment bundle on M. The dual orbibundle L^{-1} is called a *dual moment bundle*. Theorem 2.9 fails for dual moment bundles.

Example 2.13 (Vergne; cf. [22]). Let $G = \mathrm{SU}(2)$. Then $\mathfrak{t}^*_+ \cong i\mathbb{R}_+$, $\Lambda^* \cong 2\pi i\mathbb{Z}$ and $\Lambda^*_+ \cong 2\pi i\mathbb{N}$. Under these identifications the positive root $\alpha = 2\rho \in \Lambda^*_+$ corresponds to $4\pi i$. Let M be the projective line $\mathbb{C}P^1$ with ω equal to twice the standard Kähler form. The prequantum bundle on M is $L = \mathcal{O}(2)$, so that $H^0(M, L^{-1}) = \{0\}$ and $\dim H^1(M, L^{-1}) = 1$. It follows that G acts trivially on $H^1(M, L^{-1})$ and that $\mathrm{RR}(M, L^{-1}) = \mathrm{RR}(M, L^{-1})^G = -1$. On the other hand, $\Delta = \{2\rho\}$ and so M_0 is empty. Thus $\mathrm{RR}(M, L^{-1})^G \neq \mathrm{RR}\big(M_0, L_0^{-1}\big)$.

It is nevertheless possible to generalize Corollaries 2.11 and 2.12 to dual moment bundles. As is to be expected, the correct multiplicity formula involves some signs and shifts by half the sum of the positive roots.

Theorem 2.14. *Let L be a moment bundle on the compact Hamiltonian G-orbifold M. Then*

$$\mathrm{RR}(M, L^{-1}) = (-1)^{\dim \Delta} \sum_{\mu \in \Lambda^* \cap \operatorname{int} \Delta} \mathrm{RR}\big(M_\mu, (L_\mu^{\mathrm{shift}})^{-1}\big) \operatorname{Ind}_T^G \zeta_{-\mu}. \tag{2.8}$$

It follows that the support of the multiplicity function satisfies

$$\operatorname{supp} N_{L^{-1}} \subset *\big(\operatorname{int} \Delta - 2(\rho - \rho_\sigma)\big) \cap \Lambda^*_+, \tag{2.9}$$

where σ is the principal wall of M.

Here Ind_T^G is the induction functor defined in (2.6); the principal wall of M is the smallest open wall σ of the Weyl chamber such that $\Delta \subset \bar{\sigma}$; and ρ_σ denotes half the sum of the positive roots of the centralizer G_σ (so $\rho - \rho_\sigma$ is equal to the orthogonal projection of ρ onto σ). See Section 6.2 for the proof. Note that in contrast to Theorem 2.9 the summation is only over the relative interior of the moment polytope. Similar formulas hold of course for all tensor powers L^{-m}. This result may be viewed as a generalization of Ehrhart's reciprocity theorem for the number of lattice points in a convex polytope, as we shall see in Section 5.3.

It is not hard to see from (2.9) that for every $\nu \in \operatorname{supp} N_{L^{-1}}$ there exist *unique* $w \in \mathfrak{W}$ and $\mu \in \Lambda^*_+ \cap \operatorname{int} \Delta$ such that $\nu = w \odot (-\mu)$. In fact, $w = w_0 w_\sigma$ and $\mu = -w_\sigma w_0 \odot \nu$, where w_σ is the longest Weyl group element of G_σ. (See Lemma

6.9.) Consequently, only the symplectic quotient at $-w_\sigma w_0 \odot \nu$ contributes to the multiplicity at ν:

$$N_{L^{-1}}(\nu) = \mathrm{RR}\big(M_{-w_\sigma w_0 \odot \nu}, (L^{\mathrm{shift}}_{-w_\sigma w_0 \odot \nu})^{-1}\big).$$

Here are some special cases of (2.9): if Δ contains strictly dominant points, then $\sigma = \operatorname{int} \mathfrak{t}^*_+$ and $\rho_\sigma = 0$, so $\operatorname{supp} N_{L^{-1}} \subset (*\operatorname{int}\Delta - 2\rho) \cap \Lambda^*_+$. If $\Delta = \{0\}$, then $\operatorname{supp} N_{L^{-1}} \subset \{0\}$. If $G = \mathrm{SU}(3)$ and σ is the wall spanned by the fundamental weight λ_1, then $\rho_\sigma = \frac{1}{2}\alpha_2$, so $\rho - \rho_\sigma = \frac{3}{2}\lambda_1$ and $\operatorname{supp} N_{L^{-1}} \subset (*\operatorname{int}\Delta - 3\lambda_2) \cap \Lambda^*_+$. Finally if G is a torus, then $\operatorname{supp} N_{L^{-1}} \subset -\operatorname{int}\Delta$.

Example 2.15. For $G = \mathrm{SU}(2)$ we have $\operatorname{Ind}_T^G \zeta_\mu = \chi_\mu$ for $\mu \geq 0$, $\operatorname{Ind}\zeta_{-\rho} = 0$ and $\operatorname{Ind}_T^G \zeta_{-\mu} = -\chi_{\mu-2\rho}$ for $\mu \geq 2\rho$. If $\dim\Delta = 1$, then

$$\mathrm{RR}\big(M, L^{-1}\big) = \sum_{\substack{\mu \geq 2\rho \\ \mu \in \operatorname{int}\Delta}} \mathrm{RR}\big(M_\mu, (L_\mu^{\mathrm{shift}})^{-1}\big)\, \chi_{\mu-2\rho}.$$

For $\Delta = \{\mu\}$ we have $\mathrm{RR}(M, L^{-1}) = 0$ unless $\mu = 0$ or $\mu \geq 2\rho$. If $\mu = 0$, then $\mathrm{RR}(M, L^{-1}) = -\mathrm{RR}\big(M_0, (L_0^{\mathrm{shift}})^{-1}\big)$ is a constant character. If $\mu \geq 2\rho$, then

$$\mathrm{RR}(M, L^{-1}) = -\mathrm{RR}\big(M_\mu, (L_\mu^{\mathrm{shift}})^{-1}\big)\, \chi_{\mu-2\rho}.$$

In Example 2.13 $\mu = 2\rho$, $\Delta = \{2\rho\}$, M_μ is a point, and $\chi_{\mu-2\rho} = \chi_0$ is the trivial one-dimensional character, so $\mathrm{RR}(M, L^{-1}) = -1$.

3. Singular symplectic quotients

In Section 3.1 we review the local normal form theorem of [41] for quotients of Hamiltonian actions on manifolds and generalize it to actions on orbifolds. We also investigate how orbibundles descend to orbibundles on the quotients and, in easy cases, how nearby quotients are related to one another. Our treatment differs from [41] in that we work with the stratification of M by infinitesimal orbit types (which leads to a stratification of the symplectic quotient M_μ into orbifolds) rather than by orbit types (which results in a stratification of M_μ into manifolds). This is more natural from our point of view, because at generic levels of the moment map the symplectic quotient is usually an orbifold, even if the original space is smooth. Moreover, it is impossible to remove orbifold singularities by the desingularization process discussed in the next section. In Section 3.4 we apply some of our results to Delzant spaces.

3.1. **Local normal form near a stratum.**

3.1.1. *Stratifying the level set.* For every Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ let $(\mathfrak{h})$ denote the conjugacy class of $\mathfrak{h}$. The *stratum of infinitesimal orbit type* $(\mathfrak{h})$ is the subset of the Hamiltonian G-orbifold M defined by

$$M_{(\mathfrak{h})} = \{\, m \in M : \mathfrak{g}_m \text{ is } G\text{-conjugate to } \mathfrak{h} \,\}.$$

The connected components of $M_{(\mathfrak{h})}$ are suborbifolds, and the set of conjugacy classes $(\mathfrak{h})$ for which $M_{(\mathfrak{h})}$ is nonempty is locally finite. There is a unique conjugacy class $(\mathfrak{h})$ with the property that $\mathfrak{h}$ is subconjugate to every other stabilizer subalgebra. The corresponding stratum is denoted by M_{prin} and is called the *principal infinitesimal orbit type stratum.* It is open, dense and connected, and it is precisely the set of points where the moment map has maximal rank.

Let Z denote the G-invariant subset

$$Z = \Phi^{-1}(0)$$

of M. By partitioning Z into sets where the dimension of the stabilizer is constant and then partitioning further into connected components we obtain a collection $\{Z_\alpha : \alpha \in \mathfrak{A}\}$ of locally closed subsets. We define a partial order $\preccurlyeq$ on the indexing set $\mathfrak{A}$ by putting $\alpha \preccurlyeq \beta$ if and only if $Z_\alpha \subseteq \bar{Z}_\beta$. The decomposition

$$Z = \bigcup_{\alpha \in \mathfrak{A}} Z_\alpha$$

is called the *infinitesimal orbit type stratification* of Z. Every Z_α arises as a connected component of some intersection $Z \cap M_{(\mathfrak{h})}$ for some subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. We denote by $(\mathfrak{g}_\alpha)$ the conjugacy class of stabilizer subalgebras corresponding to the stratum Z_α, and by $(G_\alpha) = (\exp \mathfrak{g}_\alpha)$ the corresponding class of connected subgroups. If $\alpha \preccurlyeq \beta$, then $\mathfrak{g}_\beta$ is conjugate to a subalgebra of $\mathfrak{g}_\alpha$.

As we shall see, each Z_α is a suborbifold of M and the null-foliation of the restriction of ω to Z_α is given by the G-orbits. Thus the symplectic quotient $X = M_0 = Z/G$ inherits a decomposition

$$X = \bigcup_{\alpha \in \mathfrak{A}} X_\alpha,$$

whose pieces are symplectic orbifolds. It is shown in [41] that X has a unique open, dense and connected piece (even when Z does not intersect M_{prin}). If 0 is a quasi-regular value (see Definition 2.1), then $Z = Z_\alpha$ for some α, so X is a symplectic orbifold.

We now construct orbibundle charts for the quotient mapping $\pi\colon Z_\alpha \to X_\alpha$. Let $z \in Z_\alpha$ and put $x = \pi(z)$. The orbit Gz is isotropic because $\Phi(z) = 0$. The *symplectic slice* at z is the fibre at z of the symplectic normal bundle to Gz. This is the symplectic vector orbispace V/Γ, where Γ is the orbifold structure group of M at z and

$$V = \tilde{T}_z(Gz)^\omega / \tilde{T}_z(Gz).$$

Here the superscript ω stands for symplectic orthogonal complement, and $\tilde{T}_z(Gz)$ is the uniformized tangent space of the orbit Gz, that is the tangent space of $\phi^{-1}(Gz)$ at $\tilde{z}$ in an orbifold chart $\phi\colon \tilde{U} \to U$ around z with $\phi(\tilde{z}) = z$. Let us choose a G-invariant almost complex structure on M. This induces a Hermitian structure on V. The stabilizer $H = G_z$ does not necessarily act on V, but the extension $\hat{H}$ of H by Γ determined by the following commutative diagram with exact rows does:

$$\begin{tikzcd} \Gamma \arrow[r] \arrow[d, equal] & \hat{H} \arrow[r, two heads] \arrow[d] & H \arrow[d] \\ \Gamma \arrow[r, "\tau"] & \hat{\mathrm{U}}(V) \arrow[r, two heads] & \mathrm{U}(V/\Gamma). \end{tikzcd} \tag{3.1}$$

The bottom row is the definition of the "unitary group" $\mathrm{U}(V/\Gamma)$. Here τ denotes the unitary representation of Γ on V and $\hat{\mathrm{U}}(V)$ is the group of all $\phi \in \mathrm{U}(V)$ such that there exists a group isomorphism $f\colon \Gamma \to \Gamma$ (depending on ϕ) satisfying $\phi\big(\tau(\gamma)v\big) = \tau\big(f(\gamma)\big)\phi(v)$ for all $\gamma \in \Gamma$ and $v \in V$. Note that we do not assume Γ to act effectively on V, that is to say τ need not be injective. Clearly, $\hat{\mathrm{U}}(V)$ is a closed subgroup of the normalizer of $\tau(\Gamma)$ inside $\mathrm{U}(V)$. The action of $\hat{H}$ on V is Hamiltonian with moment map $\Phi_V\colon V \to \hat{\mathfrak{h}}^*$ given by $\langle \Phi_V(v), \xi \rangle = \frac{1}{2}\omega(\xi \cdot v, v)$. Composing Φ_V with

$$\begin{tikzcd}
\Upsilon \arrow[r, hook] \arrow[d] & \Gamma \arrow[r, two heads] \arrow[d] & \Gamma/\Upsilon \arrow[d] \\
\hat{H}^0 \arrow[r, hook] \arrow[d, two heads] & \hat{H} \arrow[r, two heads] \arrow[d, two heads] & \pi_0(\hat{H}) \arrow[d, two heads] \\
H^0 \arrow[r, hook] & H \arrow[r, two heads] & \pi_0(H).
\end{tikzcd}$$

Diagram 1. Stabilizers and orbifold structure groups

the natural isomorphism $\hat{\mathfrak{h}}^* \to \mathfrak{h}^*$ we obtain a Γ-invariant map, which descends to a moment map $\Phi_{V/\Gamma}$ for the H-action on V/Γ. Let $F(H, V/\Gamma) = G \times^H \big(\mathfrak{h}^0 \times V/\Gamma\big)$ be the symplectic orbifold defined in (A.1). (See Appendix A.) The following result is a consequence of Theorem A.1 and the isotropic embedding theorem.

Theorem 3.1 (symplectic slices, [29])**.** *Assume that $\Phi(z) = 0$. A G-invariant neighourhood of z in M is isomorphic as a Hamiltonian G-orbifold to a neighbourhood of the zero section in $F(H, V/\Gamma)$. The moment map on $F(H, V/\Gamma)$ is given by $[g, \beta, \Gamma v] \mapsto g\big(\beta + \Phi_{V/\Gamma}(\Gamma v)\big)$.* ☐

Let the identity components of H and $\hat{H}$ be H^0, resp. $\hat{H}^0$, and let their component groups be $\pi_0(H)$, resp. $\pi_0(\hat{H})$. Put $\Upsilon = \Gamma \cap \hat{H}^0$. Diagram 1, which is commutative and has exact rows and columns, summarizes the relationships among these groups. Each of them depends on the point $z \in Z$. Define the vector space W by the orthogonal splitting

$$V = V^{\hat{H}^0} \oplus W. \tag{3.2}$$

Both summands carry a natural unitary representation of $\hat{H}$.

Lemma 3.2. *The origin is a weakly regular value of Φ if and only if $W = 0$ at all points in the fibre $Z = \Phi^{-1}(0)$. It is a quasi-regular value if and only if $\Phi_W^{-1}(0) = 0$ at all points in Z.*

Proof. Consider a point z in $Z_\alpha \subset Z$. Computing in the model given by Theorem 3.1 we find

$$Z = G \times^H \Phi_{V/\Gamma}^{-1}(0) \subset G \times^H (V/\Gamma), \tag{3.3}$$

$$Z_\alpha = G \times^H \big(V^{\hat{H}^0}/\Gamma\big), \tag{3.4}$$

$$\ker \mathrm{d}\Phi_z = \tilde{T}_z(Gz)^\omega/\Gamma = \big(\tilde{T}_z(Gz) \oplus V\big)/\Gamma. \tag{3.5}$$

Note that $V^{\hat{H}^0}/\Gamma$ is contained in the zero level set of $\Phi_{V/\Gamma}$. Weak regularity means that Z is a suborbifold of M and that for all z the tangent orbispace T_zZ is equal to the vector orbispace $\ker \mathrm{d}\Phi_z$. By (3.3) and (3.5), this is equivalent to Z being an open suborbifold of $G \times^H (V/\Gamma)$, which is equivalent to $\Phi_{V/\Gamma}^{-1}(0) = V/\Gamma$. That is to say $\Phi_V = 0$, which means that $\hat{H}^0$ acts trivially on V, i. e. $W = 0$.

Quasi-regularity means that $Z = Z_\alpha$. By (3.3) and (3.4), this is equivalent to $\Phi_V^{-1}(0) = V^{\hat{H}^0}$, in other words $\Phi_W^{-1}(0) = 0$. ☐

It follows from (3.4) that the quotient $X_\alpha = Z_\alpha/G$ is isomorphic near x to the symplectic vector orbispace $V^{\hat{H}^0}\big/\hat{H} = V^{\hat{H}^0}\big/\pi_0(\hat{H})$. We conclude that X_α is a symplectic orbifold whose structure group at x is $\pi_0(\hat{H})$ and that an orbibundle chart for the map $Z_\alpha \to X_\alpha$ is given by

$$\begin{tikzcd} G/H^0 \times V^{\hat{H}^0} \arrow[r] \arrow[d, "/G"] & G \times^H \left(V^{\hat{H}^0}/\Gamma\right) \arrow[r, tail] \arrow[d, "/G"] & Z_\alpha \arrow[d, "/G"] \\ V^{\hat{H}^0} \arrow[r] & V^{\hat{H}^0}\big/\pi_0(\hat{H}) \arrow[r, tail] & X_\alpha. \end{tikzcd} \tag{3.6}$$

Here the horizontal arrows on the left are quotient maps under the action of $\pi_0(\hat{H})$ and the horizontal arrows on the right represent germs of equivariant embeddings at z (in the top row) and at x (in the bottom row). The fibre at z is the orbit G/H and the general fibre is G/H^0.

3.1.2. *Neighbourhood of a stratum.* To write a normal form for a neighbourhood of a stratum Z_α we examine the symplectic normal bundle N_α of Z_α in M. This is the G-equivariant Hermitian vector orbibundle over Z_α whose fibre at $z \in Z_\alpha$ is the vector orbispace W/Γ, where W is the Hermitian vector space defined by (3.2). From the symplectic slice theorem we obtain a G-equivariant orbibundle chart

$$\begin{tikzcd} \left(G \times^{\hat{H}^0} W\right) \times V^{\hat{H}^0} \arrow[r] \arrow[d] & G \times^H (V/\Gamma) \arrow[r, tail] \arrow[d] & N_\alpha \arrow[d] \\ G/H^0 \times V^{\hat{H}^0} \arrow[r] & G \times^H \left(V^{\hat{H}^0}/\Gamma\right) \arrow[r, tail] & Z_\alpha, \end{tikzcd} \tag{3.7}$$

where again the leftmost horizontal maps are quotient maps under $\pi_0(\hat{H})$ and the rightmost horizontal maps are germs of equivariant embeddings. The fibre of N_α over z is W/Γ and the fibre of the vertical map on the left is W/Υ at every point. (Note that (3.7) is not a *vector* orbibundle chart, as W/Υ is not a vector space.) Despite the fact that Υ depends on the point $z \in Z_\alpha$, the vector orbispace W/Υ does not. To see this, observe that the spaces $G \times^{\hat{H}^0} W \cong G \times^{H^0} (W/\Upsilon)$ are all the same because the infinitesimal representation of $\hat{\mathfrak{h}} \cong \mathfrak{h}$ on W does not depend on $z \in Z_\alpha$.

Now fix a point z_α in the orbifold Z_α, say a smooth point. We shall decorate with a subscript α each of the above groups and vector spaces evaluated at the basepoint z_α. Thus $H_\alpha = G_{z_\alpha}$, Γ_α is the orbifold structure group of M at z_α, V_α/Γ_α is the symplectic slice at z_α, $\Upsilon_\alpha = \Gamma_\alpha \cap \hat{H}^0_\alpha$, etc. Then the conjugacy class of connected subgroups associated to the stratum Z_α is $(G_\alpha) = (H^0_\alpha)$. Moreover, the general fibres of the maps $N_\alpha \to Z_\alpha$ and $Z_\alpha \to X_\alpha$ are W_α/Υ_α, resp. G/G_α. The composition of these maps is an orbifold fibration over X_α. We can easily compute the fibres and find orbibundle charts by stacking diagram (3.7) on top of diagram (3.6). The fibre at z is the associated orbibundle $G \times^H (W/\Gamma)$ and the general fibre is $G \times^{H^0} (W/\Upsilon) \cong G \times^{G_\alpha} (W_\alpha/\Upsilon_\alpha)$.

We can now build a standard model M_α for M near Z_α as follows. First we construct a principal orbibundle P_α over Z_α and then define M_α as an associated orbibundle. For $z \in Z_\alpha$ let $P_{\alpha,z}$ be the set of all smooth maps from $G \times^{G_\alpha} (W_\alpha/\Upsilon_\alpha)$

to $G \times^H (W/\Gamma)$ that factor through G-equivariant Hermitian vector orbibundle isomorphisms from $G \times^{G_\alpha} (W_\alpha/\Upsilon_\alpha)$ to $G \times^{H^0} (W/\Upsilon)$. In other words,

$$P_{\alpha,z} = \operatorname{Iso}\bigl(G \times^{G_\alpha} (W_\alpha/\Upsilon_\alpha), G \times^{H^0} (W/\Upsilon)\bigr)^G / \pi_0(\hat{H}),$$

where Iso stands for Hermitian vector orbibundle isomorphisms (that is diffeomorphisms from one space to the other that map fibres complex-linearly and isometrically to fibres). By Lemma A.3, $P_{\alpha,z}$ is a homogeneous space under the group $K(G_\alpha, W_\alpha) = N_{G\times K}(G_\alpha)/G_\alpha$ defined in (A.3). (Here we take K to be the unitary group $\mathrm{U}(W_\alpha/\Upsilon_\alpha)$, which acts on W_α/Υ_α in a Hamiltonian fashion.) We claim that $P_\alpha = \coprod_{z\in Z_\alpha} P_{\alpha,z}$ is a principal orbibundle over X_α with structure group $K_\alpha = K(G_\alpha, W_\alpha)$. Indeed, a K_α-orbibundle chart around $x = \pi(z)$ is given by

$$\begin{tikzcd} \hat{\mathcal{X}} \arrow[r] \arrow[d, "/K_\alpha"] & \mathcal{X} \arrow[r, tail] \arrow[d, "/K_\alpha"] & P_\alpha \arrow[d, "/K_\alpha"] \\ V^{\hat{H}^0} \arrow[r] & V^{\hat{H}^0}/\pi_0(\hat{H}) \arrow[r, tail] & X_\alpha, \end{tikzcd} \tag{3.8}$$

where

$$\mathcal{X} = \operatorname{Iso}\bigl(G \times^{G_\alpha} (W_\alpha/\Upsilon_\alpha), G \times^{H^0} (W/\Upsilon)\bigr)^G \times^{\pi_0(\hat{H})} V^{\hat{H}^0},$$
$$\hat{\mathcal{X}} = \operatorname{Iso}\bigl(G \times^{G_\alpha} (W_\alpha/\Upsilon_\alpha), G \times^{H^0} (W/\Upsilon)\bigr)^G \times V^{\hat{H}^0}.$$

By construction Z_α is the associated orbibundle $P_\alpha \times^{K_\alpha} (G/G_\alpha)$. Choose a principal connection on P_α and let F_α be the Hamiltonian $G\times K_\alpha$-orbifold $F(G_\alpha, W_\alpha/\Upsilon_\alpha) = G \times^{G_\alpha} (\mathfrak{g}_\alpha^0 \times W_\alpha/\Upsilon_\alpha)$ defined in (A.1). Our standard model is the associated orbibundle

$$M_\alpha = P_\alpha \times^{K_\alpha} F_\alpha \cong P_\alpha \times^{K_\alpha} \bigl(G \times^{G_\alpha} (\mathfrak{g}_\alpha^0 \times W_\alpha/\Upsilon_\alpha)\bigr). \tag{3.9}$$

By Theorem A.1 the minimal coupling form is a closed two-form on M_α and is nondegenerate in a neighbourhood of $Z_\alpha \subset M_\alpha$. By construction the symplectic normal bundle of Z_α in M_α is isomorphic to N_α. The constant rank embedding theorem implies the following result.

Theorem 3.3 (cf. [41])**.** *There exist a G-invariant open neighbourhood U_α of Z_α in M_α and an isomorphism of Hamiltonian G-orbifolds $U_\alpha \to M$ onto a neighbourhood of Z_α in M.* □

3.1.3. *Transverse structure of the singularities.* Taking symplectic quotients and identifying $F_\alpha /\!\!/ G$ with $(W_\alpha/\Upsilon_\alpha)/\!\!/G_\alpha$ as in Example A.2 we obtain a local model for the quotient X near its stratum X_α:

$$M_\alpha/\!\!/G \cong P_\alpha \times^{K_\alpha} (F_\alpha/\!\!/G) \cong P_\alpha \times^{K_\alpha} (W_\alpha/\Upsilon_\alpha)/\!\!/G_\alpha, \tag{3.10}$$

which fibres over the stratum $X_\alpha = P_\alpha/K_\alpha$ with general fibre $(W_\alpha/\Upsilon_\alpha)/\!\!/G_\alpha$. It is instructive to do this calculation directly by exhibiting Z (locally near Z_α) as a bundle over Z_α. Consider the suborbibundle S_α of M_α defined by

$$S_\alpha = P_\alpha \times^{K_\alpha} T^*(G/G_\alpha) \cong P_\alpha \times^{K_\alpha} \bigl(G \times^{G_\alpha} \mathfrak{g}_\alpha^0\bigr). \tag{3.11}$$

This space has the following properties: $S_\alpha \cap U_\alpha$ is symplectic, Z_α is coisotropic in S_α, and in fact 0 is a regular value of $\Phi|_{S_\alpha}$ and Z_α is its zero fibre. Furthermore, the standard model M_α is a symplectic vector orbibundle over S_α with general

fibre W_α/Υ_α, and the projection $M_\alpha \to S_\alpha$ maps Z onto Z_α. In other words, the following diagram commutes:

$$\begin{tikzcd} Z \arrow[r, tail] \arrow[d, tail, two heads] & M_\alpha \arrow[d, two heads] \\ Z \cap S_\alpha = Z_\alpha \arrow[r, hook] & S_\alpha. \end{tikzcd} \tag{3.12}$$

Upon dividing out the left-hand column by the action of G we obtain the fibration $M_\alpha /\!\!/ G \to X_\alpha$ of (3.10).

Theorem 3.4 (cf. [41])**.** *A neighbourhood of X_α in X is modelled by a neighbourhood of X_α in the fibre bundle*

$$P_\alpha \times^{K_\alpha} (W_\alpha/\Upsilon_\alpha)/\!\!/G_\alpha \longrightarrow X_\alpha,$$

whose general fibre is the symplectic cone $(W_\alpha/\Upsilon_\alpha)/\!\!/G_\alpha$. □

The scalar S^1-action on W_α, which is generated by the function $w \mapsto -\frac{1}{2}\|w\|^2$, induces a Hamiltonian S^1-action on the symplectic cone $(W_\alpha/\Upsilon_\alpha)/\!\!/G_\alpha$. The base of the cone is the level set at level 1, which is called the *link* of the stratum:

$$\big((\Phi_{W_\alpha}^{-1}(0) \cap S^{2r_\alpha - 1})/\Upsilon_\alpha\big)/G_\alpha,$$

where $2r_\alpha = \dim W_\alpha$ and $S^{2r_\alpha-1}$ is the unit sphere in W_α. The symplectic quotient at level 1 is the *symplectic link* $(\mathbb{P}(W_\alpha)/\Upsilon_\alpha)/\!\!/G_\alpha$. The quotient map from the link to the symplectic link is a principal S^1-orbibundle. Thus a singular symplectic quotient is locally an iterated cone, where the base of each cone is an S^1-orbibundle over a symplectic quotient of lower depth.

3.2. **Symplectic cross-sections.** Results similar to Theorems 3.3 and 3.4 hold at arbitrary levels of the moment map. One way to see this is to invoke the shifting trick. A better way is to appeal to Guillemin and Sternberg's symplectic cross-section theorem. The version we shall discuss is borrowed from [20] and [28].

Recall that every coadjoint orbit $G\mu$ of G intersects the positive Weyl chamber $\mathfrak{t}^*_+$ in exactly one point, which we denote by $q(\mu)$. Then q is a continuous quotient mapping for the coadjoint action,

$$q\colon \mathfrak{g}^* \longrightarrow \mathfrak{t}^*_+ \sim \mathfrak{g}^*/G. \tag{3.13}$$

Let A be the identity component of the centre of G and $[G, G]$ the commutator subgroup or semisimple part. Then the intersection of A and $[G, G]$ is finite and $G = A[G, G]$. Let $\mathfrak{g} = \mathfrak{a} \oplus [\mathfrak{g}, \mathfrak{g}]$ be the corresponding decomposition of the Lie algebra. All points in an open wall σ of the positive Weyl chamber have the same centralizer G_σ. The group G_σ is connected and the stratum in $\mathfrak{g}^*$ of orbit type G_σ is the saturation $G\sigma = q^{-1}(\sigma)$ of σ. We define a partial order $\preccurlyeq$ on the open walls by putting $\tau \preccurlyeq \sigma$ if τ is contained in the closure of σ. If $\tau \preccurlyeq \sigma$ then G_σ is a subgroup of G_τ, and if σ is the top-dimensional open wall $\operatorname{int} \mathfrak{t}^*_+$, then G_σ is the maximal torus T. Therefore $T \subset G_\sigma$ for all σ. We can write $G_\sigma = A_\sigma[G_\sigma, G_\sigma]$, where A_σ is the identity component of the centre of G_σ. Let $\mathfrak{g}_\sigma = \mathfrak{a}_\sigma \oplus [\mathfrak{g}_\sigma, \mathfrak{g}_\sigma]$ be the corresponding G_σ-invariant splitting of the Lie algebra. We shall identify $\mathfrak{g}^*_\sigma$ with the subspace of $\mathfrak{g}^*$ centralized by A_σ and $\mathfrak{a}^*_\sigma$ with the subspace of $\mathfrak{g}^*$ centralized by G_σ. Then $\mathfrak{g}^*$ is

an A_σ-invariant direct sum $\mathfrak{g}^* = \mathfrak{a}_\sigma^* \oplus [\mathfrak{g}_\sigma, \mathfrak{g}_\sigma]^* \oplus \mathfrak{g}_\sigma^0$. The summand $\mathfrak{a}_\sigma^*$ is equal to the linear span of the wall σ. Define

$$\mathfrak{S}_\sigma = G_\sigma \cdot \operatorname{star}\sigma \tag{3.14}$$

where $\operatorname{star}\sigma$ is the open star $\bigcup_{\tau \succcurlyeq \sigma} \tau$ of σ. Then $\mathfrak{S}_\sigma$ is a G_σ-invariant open neighbourhood of σ in $\mathfrak{g}_\sigma^*$ and it is in fact a slice for the coadjoint action at all points of σ. For example, if $\sigma = \mathfrak{a}^*$ then $\mathfrak{S}_\sigma = \mathfrak{g}^*$, and if $\sigma = \operatorname{int}\mathfrak{t}_+^*$ then $\mathfrak{S}_\sigma = \sigma$. The *symplectic cross-section* of M over σ is the subset

$$Y_\sigma = \Phi^{-1}(\mathfrak{S}_\sigma).$$

Note that $\Phi(Y_\sigma) \subset \mathfrak{g}_\sigma^*$. Let M_σ denote the G-invariant open subset GY_σ of M.

Theorem 3.5 (symplectic cross-sections)**.** *For every open wall σ of $\mathfrak{t}_+^*$ the symplectic cross-section Y_σ is a connected G_σ-invariant symplectic suborbifold of M. The action map $G \times Y_\sigma \to M$ induces a diffeomorphism $G \times^{G_\sigma} Y_\sigma \to M_\sigma$. If Y_σ is nonempty, then its saturation M_σ is open and dense in M. The G_σ-action on Y_σ is Hamiltonian with moment map $\Phi|_{Y_\sigma}$. For all $\mu \in \mathfrak{S}_\sigma$ the inclusion map $\Phi^{-1}(\mu) \hookrightarrow Y_\sigma$ induces a symplectomorphism $M_\mu \cong (Y_\sigma)_\mu$.*

The *principal wall* for M is the minimal open wall σ of the Weyl chamber such that $\Delta \subset \bar{\sigma}$. The cross-section Y_σ over it is the *principal cross-section*. The action of $[G_\sigma, G_\sigma]$ on the principal cross-section is trivial, so that it is in effect a Hamiltonian A_σ-space.

Consider the symplectic manifold $T^*G \cong G \times \mathfrak{g}^*$ on which G acts by left multiplication. The moment map is projection on the second factor and so the cross-section over σ is the $G \times G_\sigma$-manifold $G \times \mathfrak{S}_\sigma$. Theorem 3.5 tells us that $G \times \mathfrak{S}_\sigma$ is a symplectic submanifold of T^*G. (The symplectic structure can also be produced by minimal coupling.) It is the *universal σ-cross-section* in the sense that the cross-section over σ of the Hamiltonian G-orbifold M is isomorphic in a natural way to the symplectic quotient

$$Y_\sigma \cong \big(M \times (G \times \mathfrak{S}_\sigma)^-\big)/\!\!/G,$$

where the superscript "$-$" means that we replace the symplectic form with its opposite. Furthermore, the bundle $G \times^{G_\sigma} Y_\sigma$ is diffeomorpic to $(G \times \mathfrak{S}_\sigma \times Y_\sigma)/\!\!/G_\sigma$ and so acquires a symplectic form. Theorem 3.5 can now be supplemented as follows.

Addendum 3.6. *For all σ the open embedding $G \times^{G_\sigma} Y_\sigma \to GY_\sigma \subset M$ is symplectic.*

Now let μ be an arbitrary point in $\mathfrak{t}_+^*$. How to stratify the fibre $\Phi^{-1}(\mu)$? First assume that μ is fixed under the coadjoint action. This case can be reduced to the case $\mu = 0$ by replacing Φ by the equivariant moment map $\Phi - \mu$. Now consider an arbitrary point μ and let σ be the open face of the positive Weyl chamber containing μ. Observe that $\Phi^{-1}(\mu) \subset Y_\sigma$ and that G_σ fixes μ. By the results of the previous section the intersections of $\Phi^{-1}(\mu)$ with infinitesimal G_σ-orbit type strata in Y_σ are orbifolds and their quotients by G_σ are symplectic orbifolds. (We can also stratify the G-invariant subset $\Phi^{-1}(G\mu)$ into orbifolds by flowing out the strata of $\Phi^{-1}(\mu)$ by the G-action.) Therefore, the symplectic quotient $M_\mu = (Y_\sigma)_\mu$ has a natural decomposition into symplectic orbifolds and structure theorems analogous to 3.3 and 3.4 hold.

3.3. **Orbibundles and nearby quotients.** We give some applications of Theorem 3.3, namely to quotients of line orbibundles and to the variation of symplectic quotients. We also discuss a property of moment and rigid bundles.

Let L be a G-equivariant line orbibundle on M and let z be a point in $Z = \Phi^{-1}(0)$. Let Γ be the structure group of z and let $\phi\colon \tilde{U} \to U$ be an orbifold chart around z, where $\tilde{U}$ is a Γ-invariant ball about the origin in $\tilde{T}_z M$ and $\phi(0) = z$. Let $\tilde{L}$ be a line bundle on $\tilde{U}$ such that $L|_U = \tilde{L}/\Gamma$. We may assume that U is invariant under the action of the stabilizer $H = G_z$. As in (3.1) the action of H lifts to actions of groups $\hat{H}$ on $\tilde{U}$ and $\hat{H}^L$ on $\tilde{L}$. The extensions $\hat{H}$ and $\hat{H}^L$ are not necessarily isomorphic (unless Γ acts effectively), but the natural homomorphism $\beta\colon \hat{H}^L \to \hat{H}$ restricts to a covering homomorphism $(\hat{H}^L)^0 \to \hat{H}^0$, so that we may identify the Lie algebras $\hat{\mathfrak{h}}^L$, $\hat{\mathfrak{h}}$ and $\mathfrak{h}$.

Lemma 3.7. *The line orbibundle L is almost equivariantly locally trivial at z if and only if $\tilde{L}$ has an $\mathfrak{h}$-invariant section that does not vanish at 0. If either of these conditions holds, then L is almost equivariantly locally trivial in a neighbourhood of z.*

Proof. If L is almost equivariantly locally trivial at z (see Definition 2.2), then H^0 acts trivially on L_z, so that $(\hat{H}^L)^0$ acts trivially on $\tilde{L}_0$. We can then produce an $\mathfrak{h}$-invariant section $\tilde{s}$ of $\tilde{L}$ that does not vanish at 0 by starting with an arbitrary section s such that $s(0) \neq 0$ and averaging:

$$\tilde{s}(y) = \int_{(\hat{H}^L)^0} g s\big((\beta g)^{-1} y\big) \mathrm{d}g.$$

Conversely, if $\tilde{L}$ has an $\mathfrak{h}$-invariant section that does not vanish at 0, then $(\hat{H}^L)^0$ acts trivially on the fibre $\tilde{L}_0$. It follows that H^0 acts trivially on L_z.

By the same token, if $\tilde{L}$ has an $\mathfrak{h}$-invariant section that does not vanish at 0, then for all v near 0 the identity component of $\beta^{-1}(\hat{H}_v^0)$ acts trivially on the fibre $\tilde{L}_v$. This implies that G_y^0 acts trivially on L_y for all y near z. □

Now assume that L is almost equivariantly locally trivial at all points of Z. We want to construct orbibundle charts for the quotient $L_0 = (L|_Z)/G$. Let V/Γ be the symplectic slice at $z \in Z$ and let Φ_V be the moment map for the $\hat{H}$-action on V. By the symplectic slice theorem we can identify a neighbourhood of z in Z with a neighbourhood of 0 in $\Phi_{V/\Gamma}^{-1}(0)$, and a neighbourhood of $x = \pi(z)$ in $M_0 = Z/G$ with a neighbourhood of the apex of the symplectic cone $(V/\Gamma)/\!\!/H$. Let E be the restriction of L to $\Phi_{V/\Gamma}^{-1}(0)$ (with respect to the chosen embedding); then near x the quotient L_0 can be identified with the quotient E/H. Choose a nowhere vanishing $\mathfrak{h}$-invariant section s of $\tilde{L}$ over $\tilde{U}$. Then s induces a nowhere vanishing section of the induced map $\tilde{E}/(\hat{H}^L)^0 \to \Phi_V^{-1}(0)/\hat{H}^0$, where $\tilde{E}$ is the restriction of $\tilde{L}$ to $\Phi_V^{-1}(0)$. In other words, $\tilde{E}/(\hat{H}^L)^0$ is a trivial complex line bundle over the space $\Phi_V^{-1}(0)/\hat{H}^0$. Dividing out by the action of $\pi_0(\hat{H})$ on $\Phi_V^{-1}(0)/\hat{H}^0$ and by $\pi_0(\hat{H}^L)$ on $\tilde{E}/(\hat{H}^L)^0$

we obtain a commutative diagram

$$\begin{tikzcd}
\tilde{E}/(\hat{H}^L)^0 \arrow[r, "/\pi_0(\hat{H}^L)"] \arrow[d] & E/H \arrow[r, tail] \arrow[d] & L_0 \arrow[d] \\
\Phi_V^{-1}(0)/\hat{H}^0 \arrow[r, "/\pi_0(\hat{H})"] & \Phi_{V/\Gamma}^{-1}(0)/H \arrow[r, tail] & M_0,
\end{tikzcd} \tag{3.15}$$

where the horizontal maps on the right denote germs of embeddings. If 0 is a quasi-regular value, then $\Phi_V^{-1}(0)/\hat{H}^0$ is a symplectic manifold and (3.15) shows that L_0 is a line orbibundle on the symplectic orbifold M_0. If 0 is not quasi-regular, we consider an atlas on L_0 consisting of diagrams as in (3.15) to be the *definition* of an orbibundle on the singular quotient M_0. Because of the symplectic cross-section theorem these observations generalize immediately to arbitrary values of the moment map.

Proposition 3.8. *Let μ be an arbitrary value of Φ. If L is a line orbibundle on M that is almost equivariantly locally trivial at level μ, then the quotient L_ν is a line orbibundle on M_ν for all ν in a neighbourhood of μ.* □

Henceforth assume that μ is a quasi-regular value, so that the level set $\Phi^{-1}(\mu)$ consists of one single stratum and $M_\mu = \Phi^{-1}(\mu)/G$ is an orbifold. Assume also for the moment that $\mu = 0$. Then $Z = Z_\alpha$ for some $\alpha \in \mathfrak{A}$, so for all ν sufficiently close to 0 the fibre $\Phi^{-1}(\nu)$ is contained in the standard neighbourhood U_α of Theorem 3.3. We conclude that the quotient M_ν is symplectomorphic to the associated orbibundle $P_\alpha \times^{K_\alpha} (F_\alpha)_\nu$, whose general fibre is the (possibly singular) quotient $(F_\alpha)_\nu$. Furthermore, if L is a G-equivariant line orbibundle on M, then the restriction of L to U_α is equivariantly isomorphic to the pullback of $L|_Z$ to U_α, because U_α retracts equivariantly onto Z. Consequently, if L is almost equivariantly locally trivial at level 0, the orbibundle L_ν is isomorphic to the pullback of the orbibundle L_0 along the map $M_\nu \to M_0$. These statements are true at all quasi-regular values of the moment map.

Proposition 3.9. *Let μ be a quasi-regular value of Φ. For all $\nu \in \mathfrak{g}^*$ sufficiently close to μ there is a symplectic fibre orbibundle $M_\nu \to M_\mu$ whose general fibre is the symplectic quotient $(F_\alpha)_\nu$, where α indicates the infinitesimal orbit type of the single stratum intersecting $\Phi^{-1}(\mu)$. If ν is also a quasi-regular value, then the fibre $(F_\alpha)_\nu$ is an orbifold. If L is a line orbibundle on M which is almost equivariantly locally trivial at level μ, then the line orbibundle L_ν is isomorphic to the pullback of the line orbibundle L_μ via the map $M_\nu \to M_\mu$.* □

We conclude this section with two facts that were referred to in Section 2.1.

Proposition 3.10. *Weakly regular values of Φ are quasi-regular.*

It is easy to see that the converse of this statement is false. For instance, if M is the complex line with the standard symplectic form and the standard circle action, which is generated by the Hamiltonian function $z \mapsto -\frac{1}{2}|z|^2$, then 0 is a quasi-regular, but not weakly regular, value.

Proof. Obvious from Lemma 3.2 for those values of Φ that are G-fixed. The general case now follows from the symplectic cross-section theorem. □

Lemma 3.11. *Assume that the Hamiltonian G-orbifold M is compact.*

1. *If the orbibundle L is rigid, then it is almost equivariantly locally trivial everywhere.*
2. *Assume that L is a moment bundle. For m in M let $\sigma_m \in \mathfrak{g}_m^*$ denote the character defining the $(G_m)^0$-action on L_m. Then σ_m is equal to the projection of $\Phi(m) \in \mathfrak{g}^*$ onto $\mathfrak{g}_m^*$. Hence L is almost equivariantly locally trivial at level 0.*
3. *If L is a moment bundle, then for all $m \in M$ the action of the identity component of $G_m \cap \big[G, G_{\Phi(m)}\big]$ on L_m is trivial.*

Proof. Let m be an arbitrary point in M. Let H be a maximal torus of $(G_m)^0$ contained in the maximal torus T of G.

To prove 1 it suffices to show that H acts trivially on L_m. Let F be the connected component of M^H that contains m. Then the orbiweight of the H-action on $L_{m'}$ is the same for all m' in F, so it suffices to show that H acts trivially on $L_{m'}$ for some m' in F. Now F is a closed, and hence compact, T-invariant symplectic suborbifold of M, so the fixed-point set F^T is nonempty. Let $m' \in F^T$; then, because L is rigid, T acts trivially on $L_{m'}$, and *a fortiori* so does H.

For the proof of 2 let ι_m denote the inclusion of $\mathfrak{g}_m$ into $\mathfrak{g}$. Note that $\iota_m^*\Phi(m)$ is a character of $\mathfrak{g}_m$, so it suffices to prove that $\langle \iota_m^*\Phi(m), \xi\rangle = \sigma_m(\xi)$ for all ξ in the Lie algebra of H. Take $m' \in F^T$ as above. Since $\langle \Phi, \xi\rangle$ is constant on F, $\langle \iota_m^*\Phi(m), \xi\rangle = \langle \Phi(m), \xi\rangle = \langle \Phi(m'), \xi\rangle = \sigma_{m'}(\xi) = \sigma_m(\xi)$, where the third equality follows from the definition of a moment bundle.

Finally, 3 follows from 2 and the fact that $\big\langle \Phi(m), [\xi, \eta]\big\rangle = 0$ for all $\xi \in \mathfrak{g}$ and $\eta \in \mathfrak{g}_{\Phi(m)}$. ☐

3.4. **Delzant spaces I.** Delzant spaces are the symplectic counterparts of projective toric varieties. They provide an example of the theory expounded above and play a role in multiple symplectic cutting, which will be discussed in Section 5.2. See Sections 4.4 and 5.3 for further results on Delzant spaces. A *Delzant space* is a multiplicity-free connected Hamiltonian H-space, where H is a torus. (Recall that a Hamiltonian H-space is *multiplicity-free* if each of its symplectic quotients is either a point or empty. By a space we mean for the present purposes an orbifold or a stratified space that arises as a symplectic quotient of an orbifold.) Delzant [11] proved that multiplicity-free Hamiltonian H-manifolds are completely characterized by their moment polytopes and showed how to reconstruct such a manifold from its polytope. His results were extended to orbifolds by Lerman and Tolman [29]. We shall adapt their construction to polyhedra that are not necessarily simple or compact to obtain a larger class of spaces, although in this more general setting it is not known if the (labelled) polyhedron determines the space. Our version of the construction is based on symplectic cutting.

3.4.1. *Symplectic cutting.* Suppose that S^1 acts on M in a Hamiltonian fashion with moment map $\psi\colon M \to \mathbb{R}$ and that the S^1-action commutes with the G-action. Consider the diagonal action of S^1 on the product $M \times \mathbb{C}$, which has moment map $\tilde{\psi}(m, z) = \psi(m) - \frac{1}{2}|z|^2$. Here $\mathbb{C}$ is the complex line equipped with the standard cirle action and the standard symplectic structure. In [27] Lerman defines the *symplectic cut* as the symplectic quotient at level 0,

$$M_{\geq 0} = (M \times \mathbb{C})/\!\!/S^1.$$

Let $M_{>0}$ be the set of points $m \in M$ with $\psi(m) > 0$ and define $\beta\colon M_{>0} \to M \times \mathbb{C}$ by

$$\beta(m) = \big(m, \sqrt{2\,\psi(m)}\,\big). \tag{3.16}$$

The zero level set of $\tilde{\psi}$ is the disjoint union of $\psi^{-1}(0) \times \{0\}$ and the image of the map $M_{>0} \times S^1 \to M \times \mathbb{C}$ sending the pair $(m, e^{i\theta})$ to $e^{i\theta} \cdot \beta(m)$. This implies that as a set $M_{\geq 0}$ is the disjoint union of a copy of $M_{>0}$ and a copy of the reduced space $M_0 = \psi^{-1}(0)/S^1$. It implies also that if 0 is a regular value of ψ, then it is a regular value of $\tilde{\psi}$.

Proposition 3.12 ([27])**.** *The canonical embeddings*

$$\iota_0\colon M_0 \longrightarrow M_{\geq 0}, \qquad \iota_{>0}\colon M_{>0} \longrightarrow M_{\geq 0} \tag{3.17}$$

are symplectic embeddings. If 0 *is a regular value of* ψ, *then* $M_{\geq 0}$ *is a symplectic orbifold. The lifted* G*-action* $g(m, z) = (gm, z)$ *on* $M \times \mathbb{C}$ *induces a Hamiltonian* G*-action on* $M_{\geq 0}$, *whose moment map is the map* $\Phi_{\geq 0}$ *induced by the* S^1*-invariant map* $(m, e^{i\theta}) \mapsto \Phi(m)$. *In particular, the original* S^1*-action on* M *induces a Hamiltonian* S^1*-action on* $M_{\geq 0}$. *The image of its moment map* $\psi_{\geq 0}$ *is* $\psi(M) \cap \mathbb{R}_{\geq 0}$. ☐

Performing these constructions with the product $M \times \mathbb{C}^-$ instead of $M \times \mathbb{C}$ (where $\mathbb{C}^-$ is the complex line endowed with the standard circle action but the opposite of the standard symplectic form), and the moment map $\psi(m) + \frac{1}{2}|z|^2$ instead of $\tilde{\psi}$, we obtain the opposite cut space $M_{\leq 0}$, which is a Hamiltonian S^1-space whose moment map $\psi_{\leq 0}$ has image $\psi(M) \cap \mathbb{R}_{\leq 0}$.

Now let L be a $G \times S^1$-equivariant line orbibundle on M. Assume that L is almost equivariantly locally trivial at level 0 with respect to the S^1-action. The *cut bundle* is the G-equivariant orbibundle on the stratified symplectic space $M_{\geq 0}$ defined by

$$L_{\geq 0} = (\mathrm{pr}_M^* L)/\!\!/S^1.$$

where $\mathrm{pr}_M\colon M \times \mathbb{C} \to M$ is the projection onto the first factor. There are canonical isomorphisms (see [34])

$$\iota_0^* L_{\geq 0} \cong L_0, \qquad \iota_{>0}^* L_{\geq 0} \cong L|_{M_{>0}}. \tag{3.18}$$

In a similar vein we can define a cut bundle $L_{\leq 0}$ on the opposite cut space $M_{\leq 0}$.

From (3.17) and (3.18) we see that cutting is a local operation in the following sense. Suppose that the circle action is defined only on a G-invariant open neighbourhood U of Z, where Z is a closed suborbifold of M of codimension one such that $M - Z$ consists of two connected components, and that the action on U is generated by a G-invariant Hamiltonian function ψ with $\psi^{-1}(0) = Z$. The data (U, Z, ψ) are called the *cutting data*. The cut $U_{\geq 0}$ is well-defined and, letting $M_{>0}$ be the component of $M - Z$ containing $U_{>0}$, we can construct a global symplectic cut

$$M_{\geq 0} = M_{>0} \amalg_{U_{>0}} U_{\geq 0} \tag{3.19}$$

by pasting $M_{>0}$ to $U_{\geq 0}$ along the open symplectic embeddings $U_{>0} \hookrightarrow M_{>0}$ and $\iota_{\geq 0}\colon U_{>0} \hookrightarrow U_{\geq 0}$. The resulting orbifold depends only on $M_{>0}$, Z, and the germs at Z of the set U and the function ψ. The opposite cut $M_{\leq 0}$ and the cut bundles $L_{\geq 0}$ and $L_{\leq 0}$ are defined likewise.

3.4.2. *Cutting the cotangent bundle.* Let H be a k-dimensional torus and let $\Lambda = \ker \exp$ be its integral lattice. A *label* is an ordered pair consisting of a nonzero lattice vector and an arbitrary real number. Let $\mathcal{S}$ be a set of labels,

$$\mathcal{S} = \big\{(v_1, r_1), (v_2, r_2), \dots, (v_n, r_n)\big\}.$$

The polyhedron $\mathcal{P}$ *associated to* $\mathcal{S}$ is the subset of $\mathfrak{h}^*$ consisting of all points μ satisfying the inequalities

$$\langle \mu, v_i \rangle \geq r_i \qquad \text{for } i = 1, 2, \dots, n.$$

The pair $(\mathcal{S}, \mathcal{P})$ is called a *labelled polyhedron.* Clearly $\mathcal{P}$ is a convex polyhedron. It is not necessarily compact or nonempty, but it has finitely many faces. For every collection of labelling vectors $v_1, \dots, v_n$ there is a nonempty open set of parameters $(r_1, \dots, r_n)$ such that the associated polyhedron is nonempty and k-dimensional. If the r_i are rational, then $\mathcal{P}$ is rational.

Now let $A\colon \mathbb{R}^n \to \mathfrak{h}$ be the linear map sending the i-th standard basis vector e_i in $\mathbb{R}^n$ to the i-th labelling vector v_i. Since A maps the standard lattice $\mathbb{Z}^n$ to Λ, it exponentiates to a homomorphism $\bar{A}\colon T^n \to H$, where T^n is the torus $\mathbb{R}^n/\mathbb{Z}^n$. The left H-action on the cotangent bundle $T^*H \cong H \times \mathfrak{h}^*$ is Hamiltonian with moment map $\mathrm{pr}_{\mathfrak{h}^*}$. Via the homomorphism $\bar{A}$, T^n also acts on T^*H in a Hamiltonian fashion. Then T^*H inherits a T^n-moment map from H, which we translate by the vector $r = (r_1, \dots, r_n) \in (\mathbb{R}^n)^*$ to obtain the moment map $\psi_r\colon T^*H \to (\mathbb{R}^n)^*$ given by

$$\psi_r = A^* \circ \mathrm{pr}_{\mathfrak{h}^*} - r. \tag{3.20}$$

Here $A^*\colon \mathfrak{h}^* \to (\mathbb{R}^n)^*$ is the transpose map of A. The moment map for the i-th circle in T^n sends (h, η) to $(A^*\eta)_i - r_i$, where $h \in H$, $\eta \in \mathfrak{h}^*$, and $(A^*\eta)_i$ is the i-th coordinate of the vector $A^*\eta \in (\mathbb{R}^n)^*$. Recall that the moment map for the standard T^n-action on $\mathbb{C}^n$ is given by $\phi(z) = -\frac{1}{2}\big(|z_1|^2, \dots, |z_n|^2\big)$.

Definition 3.13. The *Delzant space $D_{\mathcal{S}}$ labelled by $\mathcal{S}$* is the symplectic quotient at level 0 of the product $T^*H \times \mathbb{C}^n$ with respect to the diagonal T^n-action and the moment map

$$\tilde{\psi}_r(h, \eta, z) = \psi_r(h, \eta) + \phi(z). \tag{3.21}$$

Alternatively, $D_{\mathcal{S}}$ can be thought of as the stratified symplectic space obtained by performing successive symplectic cuts on T^*H with respect to each of the n circles in T^n.

Since the H-action on T^*H commutes with the T^n-action, $D_{\mathcal{S}}$ is a Hamiltonian H-space with moment map $\Psi_{\mathcal{S}}\colon D_{\mathcal{S}} \to \mathfrak{h}^*$ induced by $\mathrm{pr}_{\mathfrak{h}^*}$. Because T^*H is multiplicity-free (with respect to the H-action), $D_{\mathcal{S}}$ is multiplicity-free as well. Because the H-moment map on T^*H is surjective, the image of $\Psi_{\mathcal{S}}$ is precisely the polyhedron $\mathcal{P}$.

Features of the labelled polyhedron are reflected in features of the associated Delzant space in an interesting way. Consider for instance an arbitrary open face $\mathcal{F}$ of $\mathcal{P}$. There are two sets of labels naturally associated with $\mathcal{F}$, namely the set $\mathcal{S}_{\mathcal{F}}$ of all labels in $\mathcal{S}$ corresponding to hyperplanes containing $\mathcal{F}$, and the set $\mathcal{S}|_{\mathcal{F}}$ of all labels giving the equations and inequalities for $\bar{\mathcal{F}}$. In other words,

$$\begin{aligned} \mathcal{S}_{\mathcal{F}} &= \big\{\, (v_i, r_i) \in \mathcal{S} : \text{the hyperplane } \langle \cdot, v_i \rangle = r_i \text{ contains } \mathcal{F} \,\big\}, \\ \mathcal{S}|_{\mathcal{F}} &= \mathcal{S} \cup \big\{\, (-v_i, -r_i) : (v_i, r_i) \in \mathcal{S}_{\mathcal{F}} \,\big\}. \end{aligned} \tag{3.22}$$

(We write an equation as a pair of inequalities to present $\bar{\mathcal{F}}$ as an intersection of half-spaces.) Let $\mathfrak{h}_{\mathcal{F}} \subset \mathfrak{h}$ be the subspace of $\mathfrak{h}$ annihilating the tangent space to $\mathcal{F}$, that is the linear span of all v_i such that $(v_i, r_i) \in \mathcal{S}_{\mathcal{F}}$, and let $H_{\mathcal{F}} = \exp \mathfrak{h}_{\mathcal{F}}$ be the corresponding subtorus. It is not hard to see that the preimage of $\mathcal{F}$ under $\Psi_{\mathcal{S}}$ is a connected component of the stratum of orbit type $H_{\mathcal{F}}$. Furthermore, there is a natural identification of the preimage of $\bar{\mathcal{F}}$ with the Delzant space associated to the set of labels $\mathcal{S}|_{\mathcal{F}}$. The result of these observations is as follows.

Proposition 3.14. *The stabilizer of every point in $D_{\mathcal{S}}$ is connected and the decomposition $\mathcal{P} = \bigcup_{\mathcal{F} \preccurlyeq \mathcal{P}} \mathcal{F}$ of $\mathcal{P}$ into open faces $\mathcal{F}$ gives rise to the decomposition of $D_{\mathcal{S}}$ into H-orbit type strata*:

$$D_{\mathcal{S}} = \bigcup_{\mathcal{F} \preccurlyeq \mathcal{P}} \Psi_{\mathcal{S}}^{-1}(\mathcal{F}). \tag{3.23}$$

The stratum $\Psi_{\mathcal{S}}^{-1}(\mathcal{F})$ is an $H/H_{\mathcal{F}}$-bundle over $\mathcal{F}$. Its closure is the Delzant space $D_{\mathcal{S}|_{\mathcal{F}}}$. Its dimension is $2 \dim \mathcal{F}$ and the dimension of $D_{\mathcal{S}}$ is $2 \dim \mathcal{P}$. The H-fixed points in $D_{\mathcal{S}}$ are the preimages of the vertices of $\mathcal{P}$. If $\mathcal{P}$ is k-dimensional, the H-action is free on $\Psi_{\mathcal{S}}^{-1}(\operatorname{int} \mathcal{P})$. □

The following statements are obvious from the construction.

Proposition 3.15. *For $t \neq 0$ let $t\mathcal{S}$ be the set of labels $\{(v_1, tr_1), \dots, (v_n, tr_n)\}$. The polyhedron associated to $t\mathcal{S}$ is the dilated polyhedron $t\mathcal{P}$, and $D_{t\mathcal{S}}$ is symplectomorphic to $D_{\mathcal{S}}$ equipped with t times its original symplectic form.*

If $r = 0$, then $\mathcal{P}$ is a cone with apex at the origin, $D_{\mathcal{S}}$ is a symplectic cone, and the moment map $\Psi_{\mathcal{S}}$ is homogeneous of degree one.

When is $D_{\mathcal{S}}$ nonsingular? To determine the orbifold stratification of $D_{\mathcal{S}}$, regarded as a symplectic quotient of the manifold $H \times \mathfrak{h}^* \times \mathbb{C}^n$ by T^n, we must calculate the stabilizers of the points in $\tilde{\psi}_r^{-1}(0)$ with respect to the T^n-action, where $\tilde{\psi}_r$ is as in (3.21). Let K be the kernel of $\bar{A}$, so that we have an exact sequence

$$K \hookrightarrow T^n \xrightarrow{\bar{A}} H.$$

Note that $\bar{A}$ is not necessarily surjective and that K is not necessarily connected. The stabilizer of $(h, \eta, z) \in H \times \mathfrak{h}^* \times \mathbb{C}^n$ is $K \cap T^n_z$, where T^n_z is the stabilizer of $z \in \mathbb{C}^n$ under the T^n-action. Consider an open face $\mathcal{F}$ of $\mathcal{P}$ and a point $(h, \eta, z) \in \tilde{\psi}_r^{-1}(0)$ such that $\eta \in \mathcal{F}$, in other words the image under $\Psi_{\mathcal{S}}$ of the orbit $T^n \cdot (h, \eta, z) \in D_{\mathcal{S}}$ is contained in $\mathcal{F}$. Then it is easy to check that T^n_z is the torus $T^n_{\mathcal{F}}$ generated by the span of all vectors $e_i \in \mathbb{R}^n$ such that the label (v_i, r_i) is in $\mathcal{S}_{\mathcal{F}}$, where $\mathcal{S}_{\mathcal{F}}$ is as in (3.22). The homomorphism $\bar{A}$ restricts to a surjective map $\bar{A}_{\mathcal{F}} \colon T^n_{\mathcal{F}} \to H_{\mathcal{F}}$, and the stabilizer of (h, η, z) is exactly the kernel $K_{\mathcal{F}}$. These groups form an exact sequence

$$K_{\mathcal{F}} \hookrightarrow T^n_{\mathcal{F}} \xrightarrow{\bar{A}_{\mathcal{F}}} \!\!\!\!\rightarrow H_{\mathcal{F}},$$

where again $K_{\mathcal{F}} = K \cap T^n_{\mathcal{F}}$ is not necessarily connected. The dimension of $T^n_{\mathcal{F}}$ is the number of labels l in the set $\mathcal{S}_{\mathcal{F}}$. The dimension of $K_{\mathcal{F}}$ is the *excess* $e_{\mathcal{S}}(\mathcal{F}) = l - \operatorname{codim} \mathcal{F}$ of the open face $\mathcal{F}$. The *excess function* of $(\mathcal{S}, \mathcal{P})$ is the step function

$$e_{\mathcal{S}} = \sum_{\mathcal{F} \preccurlyeq \mathcal{P}} e_{\mathcal{S}}(\mathcal{F})\, 1_{\mathcal{F}},$$

where $1_{\mathcal{F}}$ denotes the indicator function of $\mathcal{F}$. Evidently, $e_{\mathcal{S}}$ is upper semicontinuous on $\mathcal{P}$ and for all p the set $e_{\mathcal{S}}^{-1}(p)$ of constant excess p is a union of open faces. If $\mathcal{P}$ is k-dimensional, then $e_{\mathcal{S}}$ vanishes on the interior of $\mathcal{P}$. Now let $\{\mathcal{P}_\alpha : \alpha \in \mathfrak{A}\}$ be the collection of all connected components of all sets $e_{\mathcal{S}}^{-1}(p)$, where p ranges over $\mathbb{N}$. This decomposition of $\mathcal{P}$ is called the *excess decomposition* of $\mathcal{P}$. We have proved the first part of the following result.

Proposition 3.16. 1. *The infinitesimal T^n-orbit type stratification of the space $D_{\mathcal{S}}$ is induced by the excess decomposition $\mathcal{P} = \bigcup_\alpha \mathcal{P}_\alpha$:*

$$D_{\mathcal{S}} = \bigcup_\alpha D_{\mathcal{S},\alpha}, \tag{3.24}$$

where $D_{\mathcal{S},\alpha} = \Psi_{\mathcal{S}}^{-1}(\mathcal{P}_\alpha)$. The connected subgroup of T^n corresponding to the stratum $D_{\mathcal{S},\alpha}$ is the identity component of $K_{\mathcal{F}}$, where $\mathcal{F}$ is any of the faces in $\mathcal{P}_\alpha$. The structure group of a point p in the orbifold $D_{\mathcal{S},\alpha}$ is the component group of $K_{\mathcal{F}}$, where $\mathcal{F} \subset \mathcal{P}_\alpha$ is the face containing $\Psi_{\mathcal{S}}(p)$.

2. *The subsets $\mathcal{P}_\alpha$ and $\bar{\mathcal{P}}_\alpha$ of $\mathcal{P}$, which are the images under $\Psi_{\mathcal{S}}$ of respectively $D_{\mathcal{S},\alpha}$ and $\bar{D}_{\mathcal{S},\alpha}$, are convex. In fact, $\bar{\mathcal{P}}_\alpha$ is the closure of a single face of $\mathcal{P}$.*
3. *Assume that the excess function $e_{\mathcal{S}}$ is constant on $\mathcal{P}$. Then for every open face $\mathcal{F}$ of $\mathcal{P}$ the Delzant space $D_{\mathcal{S}|_{\mathcal{F}}}$ associated to the labelled polyhedron $(\mathcal{S}|_{\mathcal{F}}, \bar{\mathcal{F}})$ is an orbifold. In particular, $D_{\mathcal{S}}$ itself is an orbifold.*
4. *Assume that $e_{\mathcal{S}}$ is constant on $\mathcal{P}$ and that for every open face $\mathcal{F}$ the vectors in $\mathcal{S}_{\mathcal{F}}$ generate the weight lattice $\Lambda \cap \mathfrak{h}_{\mathcal{F}}$ of $H_{\mathcal{F}}$. Then the Delzant spaces $D_{\mathcal{S}|_{\mathcal{F}}}$ are manifolds. In particular, $D_{\mathcal{S}}$ itself is a manifold.*
5. *For all α the following statements are equivalent:*
   (a) *$\mathcal{P}_\alpha$ is closed;*
   (b) *$D_{\mathcal{S},\alpha}$ is closed;*
   (c) *if $\mathcal{F}_1$ and $\mathcal{F}_2$ are open faces of $\mathcal{P}$ such that $\mathcal{F}_1 \preccurlyeq \mathcal{P}_\alpha$ and $\mathcal{F}_2 \cap \bar{\mathcal{P}}_\alpha \neq \emptyset$ but $\mathcal{F}_2 \not\preccurlyeq \mathcal{P}_\alpha$, then $e_{\mathcal{S}}(\mathcal{F}_2) < e_{\mathcal{S}}(\mathcal{F}_1)$;*
   (d) *the excess function $e_{\mathcal{S}|_{\mathcal{F}}}$ of the set of labels $\mathcal{S}|_{\mathcal{F}}$ is constant, where $\mathcal{F}$ is the open face of $\mathcal{P}_\alpha$ such that $\bar{\mathcal{F}} = \bar{\mathcal{P}}_\alpha$.*

Outline of proof. Part 2 follows easily from the upper semicontinuity of $e_{\mathcal{S}}$ and the following observation: let $\mathcal{F}_1$ and $\mathcal{F}_2$ be any pair of open faces of $\mathcal{P}$. Let $\mathcal{F}_1 \vee \mathcal{F}_2$ and $\mathcal{F}_1 \wedge \mathcal{F}_2$ denote respectively the smallest open face $\mathcal{F}$ such that $\mathcal{F} \succeq \mathcal{F}_1 \cup \mathcal{F}_2$ and the interior of the intersection $\bar{\mathcal{F}}_1 \cap \bar{\mathcal{F}}_2$. Then

$$e_{\mathcal{S}}(\mathcal{F}_1 \vee \mathcal{F}_2) + e_{\mathcal{S}}(\mathcal{F}_1 \wedge \mathcal{F}_2) = e_{\mathcal{S}}(\mathcal{F}_1) + e_{\mathcal{S}}(\mathcal{F}_2).$$

For part 3, observe that the excess function is constant if and only if 0 is a quasi-regular value of $\tilde{\psi}_r$, and in this case the orbifold decomposition (3.24) consists of only one piece. To see that the spaces $D_{\mathcal{S}|_{\mathcal{F}}}$ are also orbifolds, note that for any $\mathcal{F}_1$ and $\mathcal{F}_2$ such that $\mathcal{F}_2 \preccurlyeq \mathcal{F}_1$ one has

$$e_{\mathcal{S}|_{\mathcal{F}_1}}(\mathcal{F}_2) = e_{\mathcal{S}}(\mathcal{F}_2) + l, \tag{3.25}$$

where l is the number of labels in $\mathcal{S}_{\mathcal{F}_1}$. Consequently, if $e_{\mathcal{S}}$ is constant on $\mathcal{P}$, then $e_{\mathcal{S}|_{\mathcal{F}}}$ is constant on $\bar{\mathcal{F}}$ for all $\mathcal{F}$.

The hypotheses in 4 imply that the groups $K_{\mathcal{F}}$ are all connected. Hence the one orbifold occurring in the decomposition (3.24) has trivial structure groups at all points, so $D_{\mathcal{S}}$ is a manifold. The proof that the $D_{\mathcal{S}|_{\mathcal{F}}}$ are manifolds is similar.

The equivalence of 5a and 5b is trivial and the equivalence of 5a and 5c follows from the upper semicontinuity of $e_{\mathcal{S}}$ and the convexity of $\mathcal{P}_\alpha$. The equivalence of 5a and 5d is a consequence of the equality (3.25). □

Note that the stratification (3.24) is coarser than the H-orbit type stratification (3.23), which is a stratification into manifolds. Note further that 0 is a regular value of $\tilde{\psi}_r$ if and only if the excess function is identically zero, that is to say for all open faces $\mathcal{F}$ the labelling vectors occurring in $\mathcal{S}_{\mathcal{F}}$ are linearly independent (and hence form a basis of the vector space $\mathfrak{h}_{\mathcal{F}}$). In this case the set of labels and its associated polyhedron are called *simple*. If $\mathcal{S}$ is simple, then $\mathcal{P}$ is k-dimensional and the Delzant spaces associated to all the closed faces of $\mathcal{P}$ are orbifolds. The torus T^n acts freely on $\tilde{\psi}_r^{-1}(0)$ if and only if for all open faces $\mathcal{F}$ the labelling vectors in $\mathcal{S}_{\mathcal{F}}$ are a basis of the weight lattice $\Lambda \cap \mathfrak{h}_{\mathcal{F}}$. In this case $\mathcal{S}$ and $\mathcal{P}$ are called *simply laced*.

To what extent are the set $\mathcal{S}$ and the space $D_{\mathcal{S}}$ determined by the polyhedron $\mathcal{P}$? If (v_i, r_i) is a label such that $\mathcal{P}$ is contained in the open half-space $\langle \mu, v_i \rangle > r_i$, then it can be omitted from the set of labels without affecting the structure of $D_{\mathcal{S}}$. Call $\mathcal{S}$ *minimal* if it does not contain any such labels. It is clear that every set of labels can be pruned down to a unique minimal one by dropping all redundant labels. If $\mathcal{S}$ is minimal and simple, then the labels are in one-to-one correspondence with the faces of $\mathcal{P}$ and every label is determined by the corresponding face up to multiplication by a positive integer. A complete understanding of this special situation is obtained in [29]. Multiplying a label by an integer does not affect the homeomorphism type of $D_{\mathcal{S}}$, but does affect its orbifold structure. If $\mathcal{S}$ is minimal and simply laced (the case originally considered by Delzant), then $\mathcal{S}$ and $D_{\mathcal{S}}$ are in fact uniquely determined by $\mathcal{P}$. It is not hard to see that the Delzant spaces associated to general sets of labels are also determined by the underlying polyhedra up to equivariant homeomorphism, but their classification up to isomorphism of stratified symplectic spaces appears to be unknown.

We conclude this discussion by analysing the structure of $D_{\mathcal{S}}$ transverse to the singularities. Observe that for every $\mathcal{F}$ the set $\mathcal{S}_{\mathcal{F}}$ is a set of labels for the subtorus $H_{\mathcal{F}}$. Now take any $\alpha \in \mathfrak{A}$ and let $\mathcal{F}$ be the open face such that $\mathcal{P}_\alpha \subset \bar{\mathcal{F}}$. The polyhedron $\mathcal{P}_{\mathcal{F}} \subset \mathfrak{h}_{\mathcal{F}}^*$ associated to $\mathcal{S}_{\mathcal{F}}$ has two important properties: it is a strictly convex cone with apex the point $\mathrm{pr}_{\mathfrak{h}_{\mathcal{F}}^*} \mu$, where μ is any point in $\mathcal{F}$, and $\mathcal{P}_\alpha$ has a neighbourhood in $\mathcal{P}$ that is the product of $\mathcal{P}_\alpha$ and a neighbourhood of the apex in $\mathcal{P}_{\mathcal{F}}$. By Proposition 3.15 the Delzant space associated to $\mathcal{S}_{\mathcal{F}}$ is a symplectic cone and after subtracting $\mathrm{pr}_{\mathfrak{h}_{\mathcal{F}}^*} \mu$ its moment map is homogeneous of degree one. The upshot is as follows.

Proposition 3.17. *A neighbourhood of $D_{\mathcal{S},\alpha}$ in $D_{\mathcal{S}}$ is symplectomorphic to the product of $\Psi_{\mathcal{S}}^{-1}(\mathcal{P}_\alpha)$ with the symplectic cone $D_{\mathcal{S}_{\mathcal{F}}}$, where $\mathcal{F}$ is the largest open face contained in $\mathcal{P}_\alpha$.* □

The symplectic link of $\mathcal{P}_\alpha$ is also a Delzant space. We leave it to the reader to determine its labelled polytope.

4. Partial desingularizations

If M is a nonsingular projective variety over $\mathbb{C}$ on which the complex reductive group $G^{\mathbb{C}}$ acts by projective linear transformations, then the compact group G acts on M in a Hamiltonian fashion, where the symplectic form is the imaginary part

of the Fubini-Study metric. Under the hypothesis that the set of stable points is nonempty (which amounts to the symplectic condition that the zero locus of the moment map contains regular points), Kirwan showed in [25] how to construct explicitly a "partial" desingularization of the categorical quotient $X = M/\!\!/G^{\mathbb{C}}$. It is birationally equivalent to X and possesses at worst finite-quotient singularities. The construction consists in judiciously blowing up subvarieties of M until all semistable points become stable, and subsequently dividing out by the action of $G^{\mathbb{C}}$. Kirwan pointed out that this method works also in the symplectic category if one replaces complex blowups by symplectic blowups in the sense of Gromov. However, symplectic blowups depend on a number of choices and the partial desingularization obtained by this method is not unique up to symplectomorphism. A further contrast with the algebraic case is that there is no natural way to define a blowdown map.

In Sections 4.1–4.2 we review Kirwan's desingularization process in the symplectic setting and show that the result is determined uniquely up to symplectic deformations. Deformation equivalence is much weaker than symplectomorphism, but suffices for our purpose of calculating characteristic numbers. A problem is that the blowup centres involved are not compact and are indeed not even uniquely defined. We handle this difficulty by applying a version of the constant-rank embedding theorem "with parameters" proved in Appendix A. Further we show that Kirwan's method works equally well for G-actions on orbifolds and remove the hypothesis that the zero locus of the moment map should contain a regular point. In Section 4.3 we compare her desingularization to the shift desingularizations discussed in Section 2.2. Finally in Section 4.4 we apply both methods to Delzant spaces.

4.1. **The canonical partial desingularization.**

4.1.1. *Symplectic blowing up.* Let S be a locally closed G-invariant symplectic suborbifold of M and let $\mathcal{K}$ be a G-invariant closed subset of M such that $\mathcal{K} \cap S$ is compact. It is possible to define a symplectic blowup of M along S, at least in a neighbourhood of $\mathcal{K}$. (If S is compact, we can take $\mathcal{K} = M$, but we are mainly interested in cases where S is not compact.) This is most easily accomplished in terms of symplectic cutting and involves certain auxiliary data $(j, \theta, \iota, \varepsilon)$. Here j is a G-invariant compatible complex structure on the normal bundle N of S and θ a G-invariant principal connection on the orbibundle P of unitary frames in N. To define ι and ε we note that N is isomorphic to the associated vector orbibundle $P \times^{\mathrm{U}(k)} \mathbb{C}^k$, where $2k$ is the codimension of S in M and $\mathbb{C}^k$ is Euclidean space equipped with its standard symplectic form $\frac{i}{2}\sum_{l=1}^{k} \mathrm{d}z_l \wedge \mathrm{d}\bar{z}_l$. According to Theorem A.1 the minimal coupling form on N defined by means of θ is nondegenerate in a neighbourhood N' of the zero section. By the symplectic embedding theorem there exists a G-equivariant symplectic embedding $\iota\colon N' \to M$ such that the diagram

$$\begin{tikzcd} S \arrow[d] \arrow[dr, "\subset"] & \\ N' \arrow[r, "\iota"'] & M \end{tikzcd}$$

commutes. Here the vertical arrow denotes the zero section of N. By 3 of Theorem A.1 the S^1-action on N defined by scalar multiplication on the fibres is Hamiltonian with moment function $\psi(v) = -\frac{1}{2}\|v\|^2$, where $\|\cdot\|$ denotes the fibre metric on N.

For $\delta > 0$ we have respectively the open and closed disc orbibundles and the sphere orbibundle,

$$\begin{aligned} N(\delta) &= \{\, v \in N : \psi(v) > -\delta \,\}, \\ \bar{N}(\delta) &= \{\, v \in N : \psi(v) \geq -\delta \,\}, \\ \mathbb{S}N(\delta) &= \{\, v \in N : \psi(v) = -\delta \,\}. \end{aligned}$$

Because $\mathcal{K} \cap S$ is compact, there exist $\delta > 0$ and a G-invariant open neighbourhood U' of $\mathcal{K} \cap S$ such that $\bar{N}(\delta)|_{U' \cap S}$ is contained in N' and ι embeds $\bar{N}(\delta)|_{U' \cap S}$ properly into U'. In addition, because $\mathcal{K}$ is closed there exists a G-invariant open subset U'' of M such that the union $U = U' \cup U''$ contains $\mathcal{K}$ and U'' does not intersect the image of $\bar{N}(\delta)|_{U' \cap S}$ under ι. Then $U \cap S = U' \cap S$ is a closed suborbifold of U, $\iota\big(\bar{N}(\delta)|_{U \cap S}\big)$ is a closed subset of U, and the complement in U of $\iota\big(N(\delta)|_{U \cap S}\big)$ is a suborbifold with boundary $\iota\big(\mathbb{S}N(\delta)|_{U \cap S}\big)$. Now let $0 < \varepsilon < \delta$ and put $\psi_\varepsilon(v) = \psi(v) + \varepsilon$. Clearly, 0 is a regular value of ψ_ε. Let us identify $\bar{N}(\delta)|_{S \cap U'}$ with its image under ι. Then the symplectic cut of U with respect to ψ_ε is well-defined.

Definition 4.1. The *blowup* $\mathrm{Bl}(U, S, \omega, j, \theta, \iota, \varepsilon)$, or $\mathrm{Bl}(U, S, \varepsilon)$, of U with *centre* S is the Hamiltonian G-orbifold $U_{\leq 0}$ obtained by cutting U with respect to the function ψ_ε. The *exceptional divisor* is the symplectic orbifold U_0.

Thus the orbifold $\mathrm{Bl}(U, S, \omega, j, \theta, \iota, \varepsilon)$ is obtained by excising from U the open disc orbibundle about $U \cap S$ and collapsing the orbits of the S^1-action on the bounding sphere orbibundle. The exceptional divisor embeds symplectically into the blowup. It is symplectomorphic to the total space of (the restriction to $U \cap S$ of) the orbibundle $\mathbb{P}N = P \times^{\mathrm{U}(k)} \mathbb{C}P^{k-1}$, the projectivization of N, whose general fibre is the $k-1$-dimensional complex projective space with ε times the Fubini-Study form. The blowup is diffeomorphic to the orbifold $(U - S) \coprod \mathcal{I}$ obtained by gluing together $U - S$ and the incidence relation $\mathcal{I} \subset (U \cap N') \times \mathbb{P}N|_{U \cap S}$ along the obvious map. This space is called the *null-blowup* and denoted by $\mathrm{Bl}(U, S, \omega, j, \iota, 0)$ or $\mathrm{Bl}(U, S, 0)$. It is plainly independent of the data (θ, ε). Note that unlike the symplectic blowups the null-blowup has a canonical blowdown map $\pi\colon \mathrm{Bl}(U, S, 0) \to U$.

The symplectomorphism type (in fact, the symplectic volume) of a symplectic blowup depends on the parameter ε. On the other hand, it is not hard to see that given a fixed $\varepsilon > 0$ the choice of j and θ does not affect the blowup. How the symplectic structure of the blowup depends on the embedding ι is a delicate problem and is only partly understood even in the case where S is a point. (See [31, 32].) Observe however that the blowup does not actually depend on the embedding, but only on the cutting data, that is the image of the sphere orbibundle and the distance function under the embedding. The following elementary result, which is proved in [33] for S a point, says roughly that the blowup does not depend on (j, θ, ι) as long as ε is sufficiently small. In Section 4.2 we consider what happens if ω, S and ε are allowed to vary.

Proposition 4.2. *For all triples (j_0, θ_0, ι_0) and (j_1, θ_1, ι_1) there exist $\delta > 0$ and G-invariant open neighbourhoods U_0 and U_1 of $\mathcal{K}$ such that for all $\varepsilon < \delta$ the blowups $\mathrm{Bl}(U_0, S, \omega, j_0, \theta_0, \iota_0, \varepsilon)$ and $\mathrm{Bl}(U_1, S, \omega, j_1, \theta_1, \iota_1, \varepsilon)$ are isomorphic Hamiltonian G-orbifolds.*

Proof. First we reduce the problem to the case where $j_0 = j_1$ and $\theta_0 = \theta_1$. For $i = 0$, 1, let ω_i and $\|\cdot\|_i$ denote the minimal coupling form and fibre metric on N associated

to (j_i, θ_i). By the symplectic embedding theorem we can find $\delta' < \delta$, $U \supset \mathcal{K}$ and an embedding $\upsilon\colon \big(N(\delta')|_{U\cap S}, \omega_0\big) \to \big(N(\delta)|_{U\cap S}, \omega_1\big)$ of Hamiltonian $G \times S^1$-orbifolds fixing the zero section. Since υ intertwines the S^1-moment maps, it maps the sphere orbibundles associated to the fibre metric $\|\cdot\|_0$ to those for $\|\cdot\|_1$. This means that for $\varepsilon < \delta'$ the cutting data used to define the blowup $\mathrm{Bl}(U', S, \omega, j_0, \theta_0, \iota_1', \varepsilon)$ are identical to those for the blowup $\mathrm{Bl}(U', S, \omega, j_1, \theta_1, \iota_1, \varepsilon)$, where ι_1' is the composite embedding $\iota_1 \circ \upsilon$. It follows that these two blowups are isomorphic.

It remains to find U_0' and U_1' such that $\mathrm{Bl}(U_0', S, \omega, j_0, \theta_0, \iota_1', \varepsilon)$ is isomorphic to $\mathrm{Bl}(U_1', S, \omega, j_0, \theta_0, \iota_0, \varepsilon)$ for small ε. If S is a point, the map $\iota_0^{-1} \circ \iota_1'$ is symplectically isotopic to the identity in a small enough neighbourhood of S. This is in general false, but by Lemma A.9 we can find $\delta'' < \delta'$ and a G-equivariant symplectic isotopy $H\colon N(\delta'') \times [0,1] \to N(\delta')$ fixing S, starting at $\iota_0^{-1} \circ \iota_1'$ and ending at an S^1-equivariant map. As before, this implies that H_1 preserves the sphere orbibundles and the distance function in $N(\delta'')$. In other words, the map $\iota_0 H_1 (\iota_1')^{-1}$ maps the cutting data used to define the blowup with respect to the embedding ι_1' to those for the embedding ι_0. To finish the proof it suffices show that this map can be extended to a global equivariant symplectomorphism $U_0'' \to U_1''$ of suitable open U_0'' and U_1'' containing $\mathcal{K}$. This is accomplished by applying Lemma A.8, which says that the isotopy $\iota_0 H_t (\iota_1')^{-1}$ is generated by a time-dependent Hamiltonian f_t, which in the present situation we can arrange to be G-invariant. Let χ be a G-invariant cut-off function on $\iota_1' N(\delta'')$ that is compactly supported in the fibre directions and is equal to 1 on $\iota_1' N(\delta''')$ for some $\delta''' < \delta''$. Because of the compactness of $\mathcal{K} \cap S$, on a sufficiently small open U_0'' containing $\mathcal{K}$ the Hamiltonian vector field of χf_t integrates to a globally defined Hamiltonian isotopy $F\colon U_0'' \to U'$ which restricts to $\iota_0 H_t (\iota_1')^{-1}$ on $\iota_1' N(\delta''')$. We now take U_1'' to be the image of U_0'' under F_1 and conclude that for $\varepsilon < \delta'''$ the blowup $\mathrm{Bl}(U_0'', S, \omega, j_0, \theta_0, \iota_1', \varepsilon)$ is isomorphic to $\mathrm{Bl}(U_1'', S, \omega, j_0, \theta_0, \iota_0, \varepsilon)$. ☐

Consider a G-equivariant line orbibundle L on U. Then the restriction of L to $N(\delta)$ is isomorphic to $\mathrm{pr}_S^*(L|_S)$, which is trivial in the fibre directions. Fix an isomorphism $L|_{N(\delta)} \cong \mathrm{pr}_S^*(L|_S)$ and lift the action of S^1 on $N(\delta)$ to $\mathrm{pr}_S^*(L|_S)$ by letting it act trivially on the fibres.

Definition 4.3. For $0 < \varepsilon < \delta$ the *blowup* $\mathrm{Bl}(L, S, \omega, j, \theta, \iota, \varepsilon)$, or $\mathrm{Bl}(L, S, \varepsilon)$, of L along S is the G-equivariant orbibundle $L_{\le 0}$ on $\mathrm{Bl}(U, S, \varepsilon)$ obtained by cutting L with respect to the function ψ_ε. The *null-blowup* $\mathrm{Bl}(L, S, \omega, j, \iota, 0) = \mathrm{Bl}(L, S, 0)$ is the pullback of L to $\mathrm{Bl}(U, S, 0)$.

It is easy to see that the blowup of L does not depend on the identification of $L|_{N(\delta)}$ with $\mathrm{pr}_S^*(L|_S)$. We emphasize that the blowup of a prequantum line bundle is *not* prequantizing, because the curvature form of $\mathrm{Bl}(L, S, \varepsilon)$ is degenerate along the exceptional divisor. Indeed, under a diffeomorphism $\mathrm{Bl}(U, S, 0) \to \mathrm{Bl}(U, S, \varepsilon)$ the pullback of $\mathrm{Bl}(L, S, \varepsilon)$ is isomorphic to $\mathrm{Bl}(L, S, 0)$, whose curvature is equal to the presymplectic form $\pi^*\omega$.

Analogous constructions can be carried out in the presence of an arbitrary S^1-action that is defined on an open neighbourhood of S and leaves S fixed. The weights $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ of the action are assumed to be positive. In this situation N can be written as $P \times^K \mathbb{C}^k$, where K is the centralizer in $\mathrm{U}(k)$ of the circle subgroup defined by the weights α, and P is now the orbibundle of

S^1-equivariant unitary frames. The symplectic cut of U with respect to the S^1-action is referred to as a *weighted blowup* and is denoted by $\mathrm{Bl}(U, S, \omega, j, \theta, \iota, \varepsilon, \alpha)$ or $\mathrm{Bl}(U, S, \varepsilon, \alpha)$. The exceptional divisor of a weighted blowup of $\mathbb{C}^k$ is a weighted projective space.

4.1.2. *Resolving singularities.* Suppose that 0 is not a quasi-regular value of the moment map $\Phi\colon M \to \mathfrak{g}^*$. Then the stratification of the quotient $X = M/\!\!/G$ consists of more than one piece. By performing a succession of equivariant blowups of an open neighbourhood U of $Z = \Phi^{-1}(0)$ we shall define a Hamiltonian G-orbifold $(\tilde{U}, \tilde{\omega}, \tilde{\Phi})$ such that 0 is a quasi-regular value of $\tilde{\Phi}$. The blowup centres are "the" minimal G-invariant symplectic suborbifolds containing the strata of maximal depth in Z. (If G is abelian, there is a canonical choice for the blowup centres and we can furthermore arrange for them to be closed, so that the blowups are globally defined. The reason is that for all α the closure of $M_{(\mathfrak{g}_\alpha)}$ is a symplectic suborbifold of M and that Z_α is coisotropic in $\bar{M}_{(\mathfrak{g}_\alpha)}$. The component of $\bar{M}_{(\mathfrak{g}_\alpha)}$ containing Z_α is therefore a closed minimal symplectic suborbifold around Z_α.) Following Kirwan [25] we call the symplectic quotient $\tilde{X}$ of $\tilde{U}$ the *canonical partial desingularization* of X, although the process is by no means as canonical as in the algebraic case. We investigate in Section 4.2 to what extent the result is well-defined.

Recall that the *depth* of a stratum Z_α is the largest integer i for which there exists a strictly ascending chain of strata $Z_{\alpha_0} \prec Z_{\alpha_1} \prec \cdots \prec Z_{\alpha_i}$ with $\alpha_0 = \alpha$. The depth of Z is the maximum of the depths of all its strata.

Let Z_α be a stratum of maximal depth. Then Z_α is closed and hence, because Φ is proper, Z_α and X_α are compact. Choose an embedding of an open set $U_\alpha \subset M_\alpha$ as in Theorem 3.3. Let S_α be the minimal symplectic suborbifold containing Z_α defined in (3.11). Then $U_\alpha \cap S_\alpha$ is locally closed in M and by (3.12) its intersection with Z is the closed stratum Z_α. We conclude that there exists a G-invariant open neighbourhood U of the compact set $\mathcal{K} = Z$ such that for $0 < \varepsilon < \delta$ the blowup $\mathrm{Bl}(U, S_\alpha, \varepsilon)$ is well-defined. Recall that the symplectic normal bundle of S_α in U_α is the model space M_α. A neighbourhood of the exceptional divisor in $\mathrm{Bl}(U, S_\alpha, \varepsilon)$ is therefore modelled by

$$\mathrm{Bl}(M_\alpha, S_\alpha, \varepsilon) = P_\alpha \times^{K_\alpha} \big(G \times^{G_\alpha} (\mathfrak{g}_\alpha^0 \times \mathrm{Bl}(W_\alpha/\Upsilon_\alpha, 0, \varepsilon))\big).$$

The point of blowing up is that it reduces the depth of the stratification.

Lemma 4.4. 1. *Let H be a connected Lie group acting on a vector orbispace W/Υ in a unitary fashion. Let $w \neq 0$ be a point in $\Phi_{W/\Upsilon}^{-1}(0)$ and let $[w] \in \mathbb{P}(W/\Upsilon)$ be the ray through w. Then w and $[w]$ have the same infinitesimal stabilizer,* $\mathfrak{h}_w = \mathfrak{h}_{[w]}$.

2. $\operatorname{depth} \mathrm{Bl}(M_\alpha, S_\alpha, \varepsilon)/\!\!/G = \operatorname{depth} M_\alpha/\!\!/G - 1$.

Proof. Since $[w]$ is fixed under $\mathfrak{h}_{[w]}$, there is an infinitesimal character $\sigma \in (\mathfrak{h}_{[w]})^*$ such that

$$\exp(\eta) \cdot w = e^{2\pi i \langle \sigma, \eta \rangle} w \tag{4.1}$$

for all $\eta \in \mathfrak{h}_{[w]}$. Hence $\langle \Phi_{W/\Upsilon}(w), \eta \rangle = \frac{1}{2}\omega(\eta \cdot w, w) = -\pi \langle \sigma, \eta \rangle \|w\|^2$ for all $\eta \in \mathfrak{h}_{[w]}$. Now $w \neq 0$ and $\Phi_{W/\Upsilon}(w) = 0$, so $\sigma = 0$. By (4.1) we conclude that $\mathfrak{h}_{[w]} \subset \mathfrak{h}_w$. The reverse inclusion is obvious. This proves 1.

For the proof of 2 note that the depths of $M_\alpha/\!\!/G$ and $\mathrm{Bl}(M_\alpha, S_\alpha, \varepsilon)/\!\!/G$ are equal to those of $(W_\alpha/\Upsilon_\alpha)/\!\!/G_\alpha$ and $\mathrm{Bl}(W_\alpha/\Upsilon_\alpha, 0, \varepsilon)/\!\!/G_\alpha$, respectively. Recall that the

set of G_α-fixed points in W_α/Υ_α consists of the origin only. In other words, the stratum of maximal depth in $\Phi^{-1}_{W_\alpha/\Upsilon_\alpha}(0)$ consists of the origin only. Assertion 1 is therefore tantamount to saying that the zero level set of $\mathrm{Bl}(W_\alpha/\Upsilon_\alpha, 0, \varepsilon)$ contains exactly the same infinitesimal orbit types as the zero level set of W_α/Υ_α, except for the maximal element $\mathfrak{g}_\alpha$. ☐

This lemma tells us that by successively blowing up neighbourhoods of the zero level set along minimal symplectic suborbifolds containing the strata of maximal depth in the zero level set, we arrive eventually at a Hamiltonian G-orbifold $\big(\tilde{U}, \tilde{\omega}, \tilde{\Phi}\big)$ whose zero level set has depth 0, that is to say 0 is a quasi-regular value of $\tilde{\Phi}$. The canonical partial desingularization of $X = M/\!\!/G$ is by definition the orbifold

$$\tilde{X} = \tilde{U}/\!\!/G = \tilde{Z}/G.$$

For abelian G, at each stage in the process the blowup centre is uniquely defined and closed, so if M is compact, the successive blowups are globally defined.

If L is an almost equivariantly locally trivial line orbibundle on M, then by blowing it up at each step along the way, we obtain an almost equivariantly locally trivial line orbibundle $\tilde{L}$ on $\tilde{U}$. We define the canonical desingularization of $L_0 = L/\!\!/G$ to be the line orbibundle $\tilde{L}_0 = \tilde{L}/\!\!/G$.

Note finally that by choosing appropriate symplectic cross-sections we can apply the process described above to partially resolve the singularities of M_μ for every value μ of the moment map.

4.2. **Deformation equivalence.** A *deformation* of the equivariant symplectic form (ω, Φ) on M is a smooth path of equivariant symplectic forms (ω_t, Φ_t) defined for $0 \le t \le 1$ such that $(\omega_0, \Phi_0) = (\omega, \Phi)$ and $\Phi_t^{-1}(0) = \Phi^{-1}(0)$ for all t. The endpoints (ω_0, Φ_0) and (ω_1, Φ_1) of the path are called *deformation equivalent*. A *deformation equivalence* between (M, ω, Φ) and a second Hamiltonian G-orbifold (M', ω', Φ') is a diffeomorphism F from M to M' such that (ω, Φ) is deformation equivalent to $(F^*\omega', F^*\Phi')$.

For trivial G-actions (where $\Phi = 0$) this reduces to the usual notion of a deformation or pseudo-isotopy; see [33, Ch. 6]. For the purposes of this paper its interest is first of all that it is preserved under symplectic reduction and that a deformation equivalence class determines a homotopy class of almost complex structures.

Lemma 4.5.
1. *Let (ω_t, Φ_t) be a deformation of (ω, Φ). If 0 is a quasi-regular value of Φ, then for all t it is a quasi-regular value of Φ_t, and the ω_t induce a smooth path of symplectic forms on $\Phi_t^{-1}(0)/G = \Phi^{-1}(0)/G$.*
2. *The space of all G-invariant almost complex structures on M that are compatible with some equivariant symplectic structure in the deformation equivalence class of (ω, Φ) is nonempty and path-connected.*

Proof. Part 1 follows directly from the observation that the zero level set is by definition fixed under a deformation and that its stratification depends not on the symplectic form but on the G-action alone. Part 2 follows from the fact that for every path ω_t of invariant symplectic forms on M there exists a path of invariant complex structures J_t on TM such that J_t is compatible with ω_t. ☐

The second important property is that the operation of symplectic cutting (and hence the operation of blowing up) is well-behaved with respect to deformations.

Definition 4.6. A *family of cutting data* consists of sextuples $(\omega_t, \Phi_t, U_t, Z_t, \psi_t, F_t)$ defined for $0 \le t \le 1$. Here (ω_t, Φ_t) is a path of equivariant symplectic forms (we do not require $\Phi_t^{-1}(0) = \Phi^{-1}(0)$), (U_t, Z_t, ψ_t) are G-invariant cutting data with respect to the symplectic form ω_t (see Section 3.4.1), and $F\colon U_0 \times [0,1] \to M$ is an isotopy of the open subset U_0. These data are subject to the following conditions: F_0 is the identity map of U_0, F_t is G-equivariant and maps U_0 onto U_t, and the path of G-equivariant symplectic forms $(F_t^*\omega_t, F_t^*\Phi_t)$ on U_0 is a deformation of (ω, Φ). Furthermore, F_t is to be equivariant with respect to the given S^1-actions on U_0 and U_t, and the path of S^1-equivariant symplectic forms $(F_t^*\omega_t, F_t^*\psi_t)$ on U_0 is to be a deformation of (ω, ψ_0). Finally ψ_t is required to depend smoothly on t and 0 is required to be a regular value of ψ_t for all t.

These conditions entail

$$F_t\bigl(\Phi_0^{-1}(0)\bigr) = \Phi_t^{-1}(0), \qquad F_t(Z_0) = Z_t, \qquad (F_t)_*\Xi_0 = \Xi_t,$$

where Ξ_t is the Hamiltonian vector field of ψ_t with respect to ω_t. We denote by $(M, Z_t)_{\ge 0}$ the symplectic cut of M with respect to (U_t, Z_t, ψ_t) and by $(M, Z_t)_0$ the symplectic quotient. For brevity let us denote the extension of the equivariant symplectic form (ω_t, Φ_t) to $(M, Z_t)_{\ge 0}$ also by (ω_t, Φ_t). Mark that F_t does not necessarily pull back the function ψ_t to ψ_0, but that it does map $\psi_0^{-1}\bigl([0,\infty)\bigr)$ to $\psi_t^{-1}\bigl([0,\infty)\bigr)$.

We wish to show that a family of cutting data gives rise to deformation equivalent symplectic cuts. To do this, we need first to extend F to a global G-equivariant diffeotopy $\check{F}$ of M leaving the fibre $\Phi^{-1}(0)$ invariant. Let η_t be the infinitesimal generator of F; it is a time-dependent vector field supported on the track $\mathfrak{U} = \bigcup_t U_t \times \{t\}$ of the isotopy. Let $\mathfrak{U}'$ and $\mathfrak{U}''$ be $G\times S^1$-invariant tubular neighbourhoods of the hypersurface $\bigcup_t Z_t \times \{t\}$ in $\mathfrak{U}$ such that $\mathfrak{U}'' \subset \mathfrak{U}'$, and choose a $G\times S^1$-invariant bump function $\chi\colon \mathfrak{U} \to [0,1]$ that is supported on $\mathfrak{U}'$ and identically equal to 1 on $\mathfrak{U}''$. Extend the vector field $\chi\eta_t$ by 0 to a global smooth vector field on $M \times [0,1]$. Its flow, $\check{F}$, is defined for $0 \le t \le 1$ and supported on $\mathfrak{U}$. It is clearly G-equivariant and S^1-equivariant (where the S^1-action is defined). In addition, $\check{F}$ is equal on $\mathfrak{U}''$ to the previously defined flow F, and on $\mathfrak{U}$ its trajectories are subsets of those of F, so $\check{F}$ likewise preserves the set $\Phi^{-1}(0)$.

For simplicity we ignore henceforth the distinction between F and $\check{F}$. Note that F_t maps $(M, Z_0)_{>0}$ to $(M, Z_t)_{>0}$.

Proposition 4.7. *For every t the restriction of F_t to $(M, Z_0)_{>0}$ extends uniquely to a diffeomorphism $\bar{F}_t\colon (M, Z_0)_{\ge 0} \to (M, Z_t)_{\ge 0}$. The restriction of $\bar{F}_t$ to $(M, Z_0)_0$ is equal to the map $(M, Z_0)_0 \to (M, Z_t)_0$ induced by F_t. The path $(\bar{F}_t^*\omega_t, \bar{F}_t^*\Phi_t)$ of equivariant symplectic forms on $(M, Z_0)_{\ge 0}$ is a deformation of (ω_0, Φ_0).*

Proof. Define $\tilde{F}_t$ from $(U_0 - Z_0) \times \mathbb{C}$ to $(U_t - Z_t) \times \mathbb{C}$ by

$$\tilde{F}_t(u, z) = \left(F_t(u), \left|\frac{\psi_t\bigl(F_t(u)\bigr)}{\psi_0(u)}\right|^{1/2} z\right).$$

This map is clearly S^1-equivariant and maps $\tilde{\psi}_0^{-1}(0)$ to $\tilde{\psi}_t^{-1}(0)$. (Recall $\tilde{\psi}(m, z) = \psi(m) - \frac{1}{2}|z|^2$.) We assert that it extends to a diffeomorphism from $U_0 \times \mathbb{C}$ to $U_t \times \mathbb{C}$. To show this, we observe that 0 is a regular value of the composite function $\psi_t \circ F_t\colon U_0 \to \mathbb{R}$, and therefore, in suitable local coordinates $(x_1, \ldots, x_{2n})$ about a

point in Z in which Z is given by $x_1 = 0$, it can be written as $\psi_t\big(F_t(x_1, \dots, x_{2n})\big) = a(t)x_1$ with a smooth and nowhere vanishing. It follows that

$$\tilde{F}_t(u, z) = \left(F_t(u), \left|\frac{a(t)}{a(0)}\right|^{1/2} z\right)$$

extends smoothly to the locus $\{x_1 = 0\}$. Notice also that if β_t is the embedding given by (3.16) (with ψ replaced by ψ_t), then $\tilde{F}_t\big(\beta_0(u)\big) = \beta_t\big(F_t(u)\big)$, so the maps F_t and $\tilde{F}_t|_{\tilde{\psi}_0^{-1}(0)}$ can be glued together to give a smooth map $\bar{F}_t$ from $(M, Z_0)_{\geq 0}$ to $(M, Z_t)_{\geq 0}$. The proof that the inverse of F_t extends smoothly is similar. If $z = 0$, then $\tilde{F}_t(u, 0) = \big(F_t(u), 0\big)$ for all $u \in U_0$, which implies that the restriction of $\bar{F}_t$ to $(M, Z_0)_0$ is equal to the map induced by F_t.

To show that the path $(\bar{F}_t^*\omega_t, \bar{F}_t^*\Phi_t)$ is a deformation of (ω_0, Φ_0), we need merely show that $\bar{F}_t$ maps $\Phi_0^{-1}(0)$ onto $\Phi_t^{-1}(0)$. This follows from the fact that its restriction to $(M, Z_0)_{>0}$ is equal to F_t, which sends $\Phi_0^{-1}(0) \cap (M, Z_0)_{>0}$ onto $\Phi_t^{-1}(0) \cap (M, Z_t)_{>0}$. □

Proof of Theorem 2.4. The first statement of the theorem is to be interpreted as follows: suppose we apply Kirwan's desingularization process (at the level μ) and that by making two sets of choices of the parameters involved in the process we obtain two Hamiltonian G-orbifolds $(\tilde{U}, \tilde{\omega}, \tilde{\Phi})$ and $(\tilde{U}', \tilde{\omega}', \tilde{\Phi}')$. Then there exist invariant open neighbourhoods $\check{U}$ of $\tilde{\Phi}^{-1}(G\mu)$ in $\tilde{U}$ and $\check{U}'$ of $(\tilde{\Phi}')^{-1}(G\mu)$ in $\tilde{U}'$ and a deformation equivalence $\check{U} \to \check{U}'$.

After choosing an appropriate cross-section we may assume that $\mu = 0$. Observe that the first statement of the theorem together with 1 of Lemma 4.5 implies the uniqueness of the symplectic structure on $\tilde{M}_0$ up to deformations. Combined with 2 of Lemma 4.5 this shows that the Chern classes of the tangent bundle of $\tilde{M}_0$ are well-defined. Hence the Riemann-Roch numbers of $\tilde{M}_0$ are well-defined.

It remains to prove the first statement of the theorem. The proof is by induction on the depth of Z. If $\operatorname{depth} Z = 0$ there is nothing to prove. The inductive step is taken by establishing the following two facts. Let Z_α be a stratum of maximal depth in Z. Then

1. for all minimal G-invariant symplectic submanifolds S_0 and S_1 containing Z_α and for all blowup data $(j_0, \theta_0, \iota_0, \varepsilon_0)$ and $(j_1, \theta_1, \iota_1, \varepsilon_1)$ relative to the blowup centres S_0, resp. S_1, there exist invariant open neighbourhoods U_0 and U_1 of $\Phi^{-1}(0)$ such that the blowup $\mathrm{Bl}(U_0, S_0, \omega, j_0, \theta_0, \iota_0, \varepsilon_0)$ is deformation equivalent to $\mathrm{Bl}(U_1, S_1, \omega, j_1, \theta_1, \iota_1, \varepsilon_1)$;
2. for every deformation (ω_t, Φ_t) of (ω, Φ) there exist $\delta > 0$ and a family of data $(U_t, S_t, E_t, j_t, \theta_t, \iota_t, \psi_t, F_t)$ such that the following conditions are satisfied: U_t is a G-invariant open subset containing $\Phi_t^{-1}(0) = \Phi^{-1}(0)$, S_t is a closed G-invariant minimal ω_t-symplectic submanifold of U_t containing Z_α; E_t is the ω_t-symplectic normal bundle of S_t in U_t; j_t and θ_t are resp. an invariant compatible almost complex structure and an invariant connection on E_t; ι_t is a proper G-invariant ω_t-symplectic embedding of $\bar{E}_t(\delta)$ into U_t; ψ_t is the function $-\frac{1}{2}\|\cdot\|_t^2$, where $\|\cdot\|_t$ denotes the fibre metric on E_t; and finally F is a G-equivariant isotopy $E_0(\delta) \times [0,1] \to M$ starting at the identity. We require that F_t maps $E_0(\delta)$ onto $E_t(\delta)$ and is equivariant for the S^1-actions on $E_0(\delta)$ and $E_t(\delta)$; that F_t maps $E_0(\delta) \cap \Phi^{-1}(0)$ into $\Phi_t^{-1}(0)$, and $\psi_0^{-1}(-\varepsilon)$ into $\psi_t^{-1}(-\varepsilon)$ for $\varepsilon < \delta$.

Notice that the first fact suffices to prove the theorem if $\operatorname{depth} Z = 1$. The second fact says that a deformation of the equivariant symplectic form on M gives rise to a family of cutting data in the sense of Definition 4.6. Proposition 4.7 then implies that the resulting symplectic cuts (i. e. blowups) are deformation equivalent. Combined with fact 1 this says that the relation of being deformation equivalent is preserved when going through a single stage in Kirwan's desingularization process. This completes the inductive step when the depth is greater than 1.

We now proceed to prove facts 1 and 2. Proposition 4.2 and Theorem A.11 show that for all quadruples $(S_0, j_0, \theta_0, \iota_0)$ and $(S_1, j_1, \theta_1, \iota_1)$ there exist $\delta > 0$ and G-invariant open U_0 and U_1 containing $\Phi^{-1}(0)$ such that for all $\varepsilon < \delta$ the blowup $\mathrm{Bl}(U_0, S_0, \omega, j_0, \theta_0, \iota_0, \varepsilon)$ is *isomorphic* to $\mathrm{Bl}(U_1, S_1, \omega, j_1, \theta_1, \iota_1, \varepsilon)$ as a Hamiltonian G-orbifold. Furthermore, according to (3.12) the projection $M_\alpha \to S_\alpha$ in the model space M_α preserves the zero fibre of Φ. Therefore, by Lemma A.5 the blowups $\mathrm{Bl}(U_0, S_0, \omega, j_0, \theta_0, \iota_0, \varepsilon)$ and $\mathrm{Bl}(U_0, S_0, \omega, j_0, \theta_0, \iota_0, \varepsilon_0)$ are deformation equivalent as Hamiltonian G-orbifolds, and so are the blowups $\mathrm{Bl}(U_1, S_1, \omega, j_1, \theta_1, \iota_1, \varepsilon)$ and $\mathrm{Bl}(U_1, S_1, \omega, j_1, \theta_1, \iota_1, \varepsilon_1)$. This proves fact 1.

The proof of fact 2 invokes the relative constant-rank embedding theorem, Theorem A.10. We use the notation introduced before Theorem A.10 and take $\mathfrak{M} = M \times [0,1]$ and $\mathfrak{Z} = Z_\alpha \times [0,1]$. The relative symplectic form $\omega_{\mathfrak{M}}$ is defined by $\omega_{\mathfrak{M}}|_{M\times\{t\}} = \omega_t$, so that the form $\tau = \omega_{\mathfrak{M}}|_{\mathfrak{Z}}$ has constant rank $\frac{1}{2}\dim Z_\alpha/G$ on $\mathfrak{Z}$. Let $\mathfrak{N}$ be the relative symplectic normal bundle of $\mathfrak{Z}$ in $\mathfrak{M}$. For the construction of the data $(S_t, E_t, j_t, \theta_t, \iota_t, \psi_t, F_t)$ we may assume that we are working in the standard model $\mathfrak{Y} = \mathfrak{S} \oplus \mathfrak{N}$. Let $Y_t = \mathfrak{Y}|_{\{t\}}$; then Y_t is symplectomorphic to (M, ω_t) near Z_α. Recall that $\mathfrak{S}$ is defined as $(\ker\tau)^*$ and that $\ker\tau$ is equal to the distribution tangent to the G-orbits on $Z_\alpha \times [0,1]$, which does not depend on the value of the base point in $[0,1]$. Therefore $\mathfrak{S}$ is equal to the product $S_0 \times [0,1]$, where $S_0 = \mathfrak{S}|_{Z_\alpha\times\{0\}}$. Now define S_t to be $\mathfrak{S}|_{\mathfrak{Z}\times\{t\}} = S_0 \times \{t\}$. Then S_t is independent of t as a manifold, although its symplectic form may depend on t. The symplectic normal bundle $\mathfrak{E}$ of $\mathfrak{S}$ is equal to $\mathfrak{Y}$, considered as an orbibundle over $\mathfrak{S}$. The unit interval being contractible, there exists a G-equivariant isomorphism of symplectic vector orbibundles $F\colon E_0 \times [0,1] \to \mathfrak{E}$, where $E_0 = \mathfrak{E}|_{S_0}$. Let $F_t\colon E_0 \to E_t$ be the map $E_0 \to \mathfrak{E}|_{S_t}$ defined by F; then F_t is an isomorphism of symplectic vector orbibundles covering the diffeomorphism $S_0 \to S_t = S_0 \times \{t\}$. By construction $\mathfrak{E}|_{S_t}$ is simply the ω_t-symplectic normal bundle of S_t in Y_t. Choose an invariant compatible almost complex structure j_0 and an invariant connection θ_0 on E_0 and put $j_t = (F_t^{-1})^* j_0$ and $\theta_t = (F_t^{-1})^*\theta_0$. We let ι_t be the standard embedding from E_t into Y_t and $\psi_t = -\frac{1}{2}\|\cdot\|_t^2$, where $\|\cdot\|_t$ is the fibre metric on E_t with respect to the complex structure j_t. Again by construction, F_t is a $G \times S^1$-equivariant isomorphism of Hermitian vector orbibundles and therefore maps disc bundles into disc bundles and sphere bundles into sphere bundles. Recall that by (3.12) the projection $M_\alpha \to S_\alpha$ preserves $\Phi_t^{-1}(0)$ for all t, so $\Phi_t^{-1}(0) = \Phi_{S_t}^{-1}(0) \cap \Phi_{\theta_t}^{-1}(0)$ for all t by Example A.4. Furthermore $\Phi_{S_t}^{-1}(0) = Z_\alpha$ and F_t maps θ_0 to θ_t for all t, so we conclude that F_t maps $E_0(\delta) \cap \Phi_0^{-1}(0)$ into $\Phi_t^{-1}(0)$.

It remains to define the open neighbourhoods U_t of $\Phi^{-1}(0)$. We do this by starting with an invariant open U_0 containing $\Phi^{-1}(0)$ such that the embedding $\bar{E}_0(\delta) \to U_0$ is proper. We extend the infinitesimal generator of F_t to a globally defined time-dependent vector field on U_0 by means of a suitable cut-off function. Because the level set $\Phi^{-1}(0)$ is compact, after shrinking U_0 and δ if necessary the

resulting vector field is integrable for $0 \leq t \leq 1$. As a result we obtain an equivariant isotopy $\check{F}\colon U_0 \times [0,1] \to M$ preserving $\Phi^{-1}(0)$; and we put $U_t = F_t(U_0)$. $\square$

4.3. **Shift desingularizations.** The process delineated in Section 4.1.2 is usually not the most economical method for resolving the singularities of the quotient. A simpler desingularization is often obtained by shifting the value of the moment map to a nearby quasi-regular value. Let Δ_i be one of the open chambers of the moment polyhedron as in (2.7) and assume μ is in its closure. Since Φ has maximal rank on Δ_i, all $\nu \in \Delta_i$ are quasi-regular values of Φ and all M_ν have the same dimension. If this dimension is the same as that of M_μ, the M_ν are called *shift desingularizations* of X. We discuss briefly the relationship between M_ν and the canonical desingularization $\tilde{M}_\mu$ (even if they do not have the same dimension). We say that two symplectic orbifolds Q_0 and Q_1 are *related* by a weighted symplectic blowup (resp. blowdown) if Q_1 is deformation equivalent to a weighted blowup (resp. blowdown) of Q_0 at a closed symplectic suborbifold.

Theorem 4.8. *Let Δ_i be an open chamber of Δ such that μ is in the closure of Δ_i. There exists a symplectic fibre orbibundle E over $\tilde{M}_\mu$ with the property that for all ν in Δ_i the symplectic orbifold M_ν is related to E by a sequence of weighted symplectic blowups and blowdowns. The general fibre of E is a generic reduced space of the space $F(G_\alpha, W_\alpha/\Upsilon_\alpha)$, where α is the infinitesimal orbit type of the open stratum of Z.*

Sketch of proof. After applying the symplectic cross-section theorem we may assume that $\mu = 0$. By the implicit function theorem the symplectic quotients M_ν for $\nu \in \Delta_i$ are all deformation equivalent, so it suffices to prove the theorem for ν close to 0. If ν is sufficiently close to 0 we can arrange, by choosing the parameters ε in the desingularization process small enough, that ν is contained in the image of $\tilde{U}$ under $\tilde{\Phi}$ and lies outside the images of the exceptional divisors arising in the process. Then ν is a generic value of $\tilde{\Phi}$ and the quotients M_ν and $\tilde{U}_\nu$ are symplectomorphic. We now apply Proposition 3.9 to the space $\tilde{U}$ and conclude that for small generic values ν' of $\tilde{\Phi}$ the quotient $E = \tilde{U}_{\nu'}$ is a fibre orbibundle over $\tilde{M}_0$ with general fibre $F(G_\alpha, W_\alpha/\Upsilon_\alpha)_{\nu'}$. According to the symplectic cross-section theorem, the preimage $Y = \tilde{\Phi}^{-1}\bigl(\mathrm{int}(\tilde{\Phi}(\tilde{U}) \cap \mathfrak{t}^*_+)\bigr)$ is a Hamiltonian T-orbifold and $Y_\nu \cong M_\nu$ and $Y_{\nu'} \cong E$. A result of Guillemin and Sternberg [19] now says that Y_ν is related to $Y_{\nu'}$ by a sequence of weighted blowups and blowdowns. $\square$

4.4. **Delzant spaces II.** Let $\mathcal{S} = \bigl\{(v_1, r_1), (v_2, r_2), \dots, (v_n, r_n)\bigr\}$ be a set of labels for the torus T. Assume that the polyhedron $\mathcal{P}$ associated to $\mathcal{S}$ is nonempty. We can apply both desingularization processes to the Delzant space $D_{\mathcal{S}}$ defined in Section 3.4. Let us first do the canonical desingularization. By Proposition 3.16 a closed stratum $D_{\mathcal{S},\alpha}$ in $D_{\mathcal{S}}$ corresponds to a piece $\mathcal{P}_\alpha$ in the excess decomposition of $\mathcal{P}$ that is the closure of a single open face $\mathcal{F}$. Let $\mathcal{S}_{\mathcal{F}}$ be its associated set of labels as defined in (3.22). Blowing up at $D_{\mathcal{S},\alpha}$ has the effect of adding one label to the set $\mathcal{S}$:

$$\tilde{\mathcal{S}}_{\varepsilon_1} = \bigl\{(v_1, r_1), \dots, (v_n, r_n), (v_{n+1}, r_{n+1} + \varepsilon_1)\bigr\},$$

where

$$v_{n+1} = \sum_{(v_i, r_i) \in \mathcal{S}_{\mathcal{F}}} v_i, \qquad r_{n+1} = \sum_{(v_i, r_i) \in \mathcal{S}_{\mathcal{F}}} r_i$$

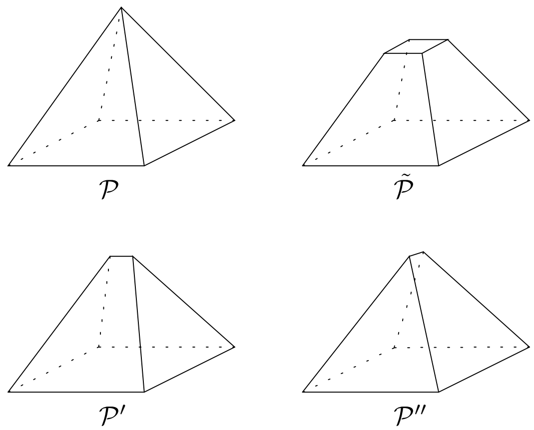

Diagram 2. Egyptian pyramid and desingularizations

and $\varepsilon_1 > 0$ is sufficiently small. The excess decomposition of the polyhedron $\tilde{\mathcal{P}}_{\varepsilon_1}$ associated to $\tilde{\mathcal{S}}_{\varepsilon_1}$ has one piece less than that of $\mathcal{P}$. Iterating this process we obtain eventually a labelled polyhedron $\big(\tilde{\mathcal{S}}_\varepsilon, \tilde{\mathcal{P}}_\varepsilon\big)$ with constant excess function, where ε denotes a vector with small positive entries. The canonical desingularization of $D_{\mathcal{S}}$ is then the Delzant space $\tilde{D}_{\mathcal{S}} = D_{\tilde{\mathcal{S}}_\varepsilon}$.

Shift desingularization has the effect of replacing $\mathcal{S}$ by a set of labels

$$\mathcal{S}_\eta = \big\{(v_1, r_1 + \eta_1), (v_2, r_2 + \eta_2), \dots, (v_n, r_n + \eta_n)\big\}, \tag{4.2}$$

where η is a small vector chosen in such a way that the associated polyhedron $\mathcal{P}_\eta$ is nonempty and has constant excess function. The shape of $\mathcal{P}_\eta$ depends on the choice of η, but the directions of the faces of codimension one do not.

See Diagram 2 for the canonical desingularization $\tilde{\mathcal{P}}$ and two different shift desingularizations $\mathcal{P}'$ and $\mathcal{P}''$ of the Egyptian pyramid. The manifold X corresponding to the truncated pyramid $\tilde{\mathcal{P}}$ is a $\mathbb{C}P^1$-bundle over $\mathbb{C}P^1 \times \mathbb{C}P^1$. The section at infinity Y is a product $Y' \times Y''$, where Y' and Y'' are copies of $\mathbb{C}P^1$, and the manifolds corresponding to $\mathcal{P}'$ and $\mathcal{P}''$ are obtained by blowing down X at Y', resp. Y''. The space corresponding to $\mathcal{P}$ is obtained by blowing down X at the divisor Y.

5. The abelian case

In this section we prove the results stated in Section 2 in the case of abelian group actions. In Section 5.1 the setting is that of an almost complex orbifold $\mathcal{M}$ with an arbitrary equivariant line orbibundle $\mathcal{L}$. Here the Atiyah-Segal-Singer fixed-point formula for the equivariant index yields imprecise, but useful, qualitative information about the multiplicity diagram of $\mathcal{L}$. The discussion is inspired by that in [13], which in turn bears great similarity to the arguments of [26, 2, 6]. We deduce the first part of Theorem 2.7 (for general groups) as an immediate corollary. We then specialize to the setting of Hamiltonian actions in Section 5.2. The result of Section 5.1 furnishes precise information on the Riemann-Roch numbers of symplectic quotients at certain vertices of the moment polytope. A cut-and-paste argument involving symplectic cutting and a gluing formula for the equivariant index then produces formulæ for the Riemann-Roch numbers of symplectic quotients at other

points. In preparation for Section 6 we then generalize our results to "asymptotic" moment bundles. Finally in Section 5.3 we apply them to Delzant spaces, where they lead to a counting formula for lattice points in rational polytopes.

5.1. **Almost complex T-orbifolds.** Let $\mathcal{M}$ be a *compact* almost complex T-orbifold and $\mathcal{L} \to \mathcal{M}$ an arbitrary T-equivariant line orbibundle. As in Section 2.1, let $\Lambda = \ker \exp$ denote the integral lattice of T, $\mathrm{RR}(\mathcal{M}, \mathcal{L}) \in \mathrm{Rep}\, T$ the equivariant index of $\bar{\partial}\!\!\!/_{\mathcal{L}}$, and $N_{\mathcal{L}} \colon \Lambda^* \to \mathbb{Z}$ the multiplicity function of $\mathcal{L}$. For all components F of the fixed-point set $\mathcal{M}^T$, let $\sigma_F \in \Lambda^* \otimes \mathbb{Q}$ be the orbiweight of the T-action on $\mathcal{L}|_F$. The *weight polytope* $\Delta_{\mathcal{L}}$ of $\mathcal{L}$ is the rational convex polytope

$$\Delta_{\mathcal{L}} = \mathrm{hull}\bigl\{\sigma_F : F \subset \mathcal{M}^T\bigr\}.$$

Let N_F be the normal bundle of F in $\mathcal{M}$ and let $\alpha_{jF} \in \Lambda^* \otimes \mathbb{Q}$ be the orbiweights for the action of T on N_F, where $j = 1, 2, \ldots, \operatorname{codim}_{\mathbb{C}} F$. Denote by $\mathcal{C}_F$ be the rational cone in $\mathfrak{t}^*$ spanned by the $-\alpha_{jF}$, and by $\check{C}_F$ its dual cone

$$\check{\mathcal{C}}_F = \{\, \xi \in \mathfrak{t} : \langle \alpha_{jF}, \xi \rangle \leq 0 \text{ for all } j \,\}.$$

The data $\Delta_{\mathcal{L}}$ and $\mathcal{C}_F$ record certain information on the multiplicity diagram of $\mathcal{L}$. A vector $\xi \in \mathfrak{t}$ is called *generic* if the one-parameter subgroup generated by ξ has the same fixed points as T. This is equivalent to $\langle \alpha_{jF}, \xi \rangle \neq 0$ for all F and all j. For instance, ξ is generic if its one-parameter subgroup is dense in T.

Theorem 5.1. 1. *The support of the multiplicity function is contained in the weight polytope.*
2. *Let μ be a vertex of the weight polytope. Choose a generic ξ in $\mathfrak{t}$ such that $\langle \mu - \sigma_F, \xi \rangle < 0$ for all F with $\sigma_F \neq \mu$. Then*
$$N_{\mathcal{L}}(\mu) = \sum_{\substack{F \\ \sigma_F = \mu \\ \xi \in \check{C}_F}} \mathrm{RR}(F, \mathcal{L}|_F). \tag{5.1}$$
3. *If H is a compact connected Lie group acting on $\mathcal{M}$ and $\mathcal{L}$, and the action of H commutes with that of T and preserves the almost complex structure on $\mathcal{M}$, then* (5.1) *holds as an equality of virtual characters of H.*

The support of $N_{\mathcal{L}}$ is by definition contained in the weight lattice, so we have in fact $\operatorname{supp} N \subset \Lambda^* \cap \Delta_{\mathcal{L}}$. This implies that if the vertex μ under 2 is not integral, then $N_{\mathcal{L}}(\mu) = 0$, so the right-hand side of (5.1) vanishes.

Proof. We assume first that $\mathcal{M}$ has no orbifold singularities. According to the Atiyah-Segal-Singer equivariant index formula the character is equal to a sum

$$\mathrm{RR}(\mathcal{M}, \mathcal{L}) = \sum_F \chi_F \tag{5.2}$$

over all fixed-point components F of M. The functions χ_F are for generic $\xi \in \mathfrak{t}$ given by

$$\chi_F(\exp \xi) = e^{2\pi i \langle \sigma_F, \xi \rangle} \int_F \frac{\mathrm{Ch}(\mathcal{L}|_F)\, \mathrm{Td}(F)}{\mathrm{D}_T(N_F, \xi)}. \tag{5.3}$$

Here $\mathrm{Ch}(\mathcal{L}|_F)$ is the Chern character of $\mathcal{L}|_F$, $\mathrm{Td}(F)$ is the Todd class of F, and $\mathrm{D}_T(N_F, \xi)$ is the T-equivariant characteristic class

$$\mathrm{D}_T(N_F, \xi) = \prod_j \bigl(1 - \exp(-2\pi i \langle \alpha_{jF}, \xi \rangle - c_{jF})\bigr) \tag{5.4}$$

of N_F. The c_{jF} are the virtual Chern roots of N_F, that is to say, $c_{jF} = c_1(N_{jF})$ if N_F decomposes into a sum $\bigoplus_j N_{jF}$ of T-equivariant line bundles where T acts with weight α_{jF} on the jth summand. Because (5.4) is symmetric in the c_{jF}, $\mathrm{D}_T(N_F, \xi)$ is well-defined by the splitting principle.

The character $\mathrm{RR}(\mathcal{M}, \mathcal{L})\colon T \to \mathbb{C}$ extends to a holomorphic function on the complexified torus $T^{\mathbb{C}}$ and the χ_F extend to rational functions on $T^{\mathbb{C}}$. Substitute $-it\xi$ for ξ in (5.3) and (5.4), where t is real, and put $x = e^{2\pi t}$. Notice that (5.4) has a finite limit as $x \to 0$. This limit is 1 if $\langle \alpha_{jF}, \xi\rangle < 0$ for all j, and 0 otherwise. Applying the Riemann-Roch Theorem to F and $\mathcal{L}|_F$ we find that as $x \to 0$

$$\chi_F\bigl(\exp(-it\xi)\bigr) = \begin{cases} x^{\langle \sigma_F, \xi\rangle} \,\mathrm{RR}(F, \mathcal{L}|_F) + o\bigl(x^{\langle \sigma_F,\xi\rangle}\bigr) & \text{if } \langle \alpha_{jF}, \xi\rangle < 0 \text{ for all } j, \\ o\bigl(x^{\langle \sigma_F,\xi\rangle}\bigr) & \text{otherwise.} \end{cases} \tag{5.5}$$

Compare this estimate to the following expression obtained from (2.1):

$$\mathrm{RR}(\mathcal{M}, \mathcal{L})\bigl(\exp(-it\xi)\bigr) = \sum_{\nu \in \Lambda^*} N_{\mathcal{L}}(\nu) x^{\langle \nu, \xi\rangle}. \tag{5.6}$$

Assume that μ is not contained in the weight polytope $\Delta_{\mathcal{L}}$. Then we can select a generic ξ such that $\langle \mu - \sigma_F, \xi\rangle < 0$ for all F, i. e. $\langle \mu, \xi\rangle < \min_F \langle \sigma_F, \xi\rangle$. According to (5.2) and (5.5), $\mathrm{RR}(M, L)\bigl(\exp(-it\xi)\bigr) = O\bigl(x^{\min_F \langle \sigma_F, \xi\rangle}\bigr)$ as $x \to 0$, so all terms in (5.6) with exponent strictly less than $\min_F \langle \sigma_F, \xi\rangle$ vanish, and therefore $N_{\mathcal{L}}(\mu) = 0$. This proves 1.

If μ is a vertex of $\Delta_{\mathcal{L}}$, we can choose a generic ξ such that $\langle \mu - \sigma_F, \xi\rangle < 0$ for all F with $\sigma_F \neq \mu$, and in addition $\langle \nu, \xi\rangle \neq \langle \mu, \xi\rangle$ for those (finitely many) $\nu \in \Lambda^*$ for which $N_{\mathcal{L}}(\nu) \neq 0$. If $\langle \nu, \xi\rangle < \langle \mu, \xi\rangle$, then $\langle \nu - \sigma_F, \xi\rangle < 0$, so ν is not in $\Delta_{\mathcal{L}}$ and therefore $N_{\mathcal{L}}(\nu) = 0$ by 1. This implies that $\langle \nu, \xi\rangle > \langle \mu, \xi\rangle$ whenever $\nu \neq \mu$ and $N_{\mathcal{L}}(\nu) \neq 0$, and therefore, by (5.6), $\mathrm{RR}(M, L)\bigl(\exp(-it\xi)\bigr) = N_{\mathcal{L}}(\mu) x^{\langle \mu, \xi\rangle} + o\bigl(x^{\langle \mu,\xi\rangle}\bigr)$ as $x \to 0$. By (5.2) and (5.5), the coefficient of the term of order $\langle \mu, \xi\rangle$ in the character is equal to the sum of the $\mathrm{RR}(F, \mathcal{L}|_F)$ over all F with the property that $\mu = \sigma_F$ and $\langle \alpha_{jF}, \xi\rangle < 0$ for all $j = 1, 2, \dots, \operatorname{codim}_{\mathbb{C}} F$. This proves 2.

In the presence of orbifold singularities and the action of an additional group H we invoke the results of [5, 44, 12], according to which $\mathrm{RR}(\mathcal{M}, \mathcal{L})$, considered as an element of $\mathrm{Rep}(T \times H)$, is given by a sum (5.2), where

$$\chi_F(\exp \xi, \exp \eta) = e^{2\pi i \langle \sigma_F, \xi\rangle} \int_{\tilde{F}} \frac{1}{d_{\tilde{F}}} \frac{\mathrm{Ch}_H^{\tilde{F}}\bigl(\tilde{\mathcal{L}}|_{\tilde{F}}, \eta\bigr)\, \mathrm{Td}_H\bigl(\tilde{F}, \eta\bigr)}{\mathrm{D}_H^{\tilde{F}}(N_{\tilde{F}}, \eta)\, \mathrm{D}_{T\times H}^{\tilde{F}}\bigl(\tilde{N}_{\tilde{F}}, \xi, \eta\bigr)}$$

for $\eta \in \mathfrak{h}$ sufficiently small. Here Ch_H etc. are the H-equivariant counterparts of the characteristic classes considered above, $\tilde{F}$ is the "unwrapping" of the orbifold F, and $d_{\tilde{F}}$ is its multiplicity, which is a locally constant function on $\tilde{F}$. See [34] for a detailed discussion. Our previous argument goes through with trivial modifications and the upshot is that 1 and 2 hold for orbifolds and that 2 holds H-equivariantly. □

Proof of part 1 of Theorem 2.7. Note first that the result for arbitrary G follows from the case where G is abelian. The weight polytope of a rigid orbibundle is by definition $\{0\}$, so the abelian case is an immediate consequence of Theorem 5.1. □

5.2. **Hamiltonian T-orbifolds.** In this section (M, ω, Φ) denotes a compact connected Hamiltonian T-orbifold and L a T-equivariant line orbibundle on M. In this situation we have two polytopes, namely the moment polytope $\Delta = \Phi(M)$ and the

weight polytope Δ_L, and Theorem 5.1 yields information on the index $\mathrm{RR}(M_\mu, L_\mu)$ for certain vertices μ of Δ. For those bundles L for which there is a simple relationship between the polytopes Δ and Δ_L, namely rigid, moment, and dual moment bundles, $\mathrm{RR}(M_\mu, L_\mu)$ can then be calculated when μ is not a vertex of Δ by means of multiple symplectic cutting and the gluing formula, which we review in Section 5.2.1. In Section 5.2.2 we put these ingredients together to obtain proofs of the remaining theorems of Section 2 for abelian groups.

5.2.1. *Multiple symplectic cutting.* Let $\mathcal{S}$ be a set of labels in $\mathfrak{t}^*$ and let $\mathcal{P}$ be its associated polyhedron. Assume that the excess function of $(\mathcal{S}, \mathcal{P})$ is constant. Then the Delzant space $D_{\mathcal{S}}$ is an orbifold by Proposition 3.16 and the Hamiltonian T-orbifold $D_{-\mathcal{S}}$ is symplectomorphic to $D_{\mathcal{S}}$ with the opposite symplectic form by Proposition 3.15. The *symplectic cut of M with respect to $\mathcal{P}$* is the Hamiltonian T-space

$$M_{\mathcal{P}} = (M \times D_{-\mathcal{S}})/\!\!/T \tag{5.7}$$

obtained by reduction at 0 with respect to the diagonal T-action and the moment map $\Phi \times -\Psi_{\mathcal{S}}$. There are several alternative ways to think of $M_{\mathcal{P}}$. Firstly, by Definition 3.13 and reduction in stages,

$$M_{\mathcal{P}} = (M \times T^*T \times \mathbb{C}^n)/\!\!/(T^n \times T) = (M \times \mathbb{C}^n)/\!\!/T^n.$$

In particular, the Delzant space $D_{\mathcal{S}}$ itself is equal to the symplectic cut of T^*T with respect to $\mathcal{P}$. Secondly, $M_{\mathcal{P}}$ is the space obtained by performing successive symplectic cuts on M with respect to each of the labels in $\mathcal{S}$. Lastly, as a topological space it is equal to the inverse image $\Phi^{-1}(\mathcal{P})$ in which, for each open face $\mathcal{F}$ of $\mathcal{P}$, one divides out the preimage $\Phi^{-1}(\mathcal{F})$ by the $T_{\mathcal{F}}$-action. Thus we have a decomposition

$$M_{\mathcal{P}} = \bigcup_{\mathcal{F} \preccurlyeq \mathcal{P}} \Phi^{-1}(\mathcal{F})/T_{\mathcal{F}}. \tag{5.8}$$

We designate the moment map for the T-action on $M_{\mathcal{P}}$ by $\Phi_{\mathcal{P}}$. Its image is equal to $\Delta \cap \mathcal{P}$.

Definition 5.2. The pair $(\mathcal{S}, \mathcal{P})$ is *admissible* or *T-admissible* with respect to M if $\mathcal{S}$ has constant excess and the reduction in (5.7) is regular.

The reduction being regular is equivalent to $\mathfrak{t}_m \cap \mathfrak{t}_x = \{0\}$ for all (m, x) in $M \times D_{-\mathcal{S}}$ such that $\Phi(m) = \Psi_{\mathcal{S}}(x)$. If $\mathcal{F}$ is the open face of $\mathcal{P}$ that contains $\Psi_{\mathcal{S}}(x)$, then $\mathfrak{t}_x = \mathfrak{t}_{\mathcal{F}}$ by Proposition 3.14, so admissibility amounts to the condition

$$\mathfrak{t}_m \cap \mathfrak{t}_{\mathcal{F}} = \{0\} \qquad \text{for all } m \in M \text{ and all } \mathcal{F} \preccurlyeq \mathcal{P} \text{ such that } \Phi(m) \in \mathcal{F}. \tag{5.9}$$

This has two consequences: firstly, because $\mathfrak{t}_{\mathcal{F}}$ depends only on the face $\mathcal{F}$, admissibility depends only the polyhedron $\mathcal{P}$ and not on the set of labels defining it; and secondly, if $\mathcal{P}$ is admissible, then for every open face $\mathcal{F}$, 0 is a regular value of the moment map $\Phi \times -\Psi_{\mathcal{S}_{\mathcal{F}}}$ on the Hamiltonian T-orbifold $M \times D_{\mathcal{S}_{\mathcal{F}}}$. In other words, every closed face of $\mathcal{P}$ is admissible as well.

Example 5.3. If $T = S^1$, then $\mathfrak{t} = i\mathbb{R}$ and $\Lambda = 2\pi i\mathbb{Z}$. Let $\mathcal{S} = \{(v, 0)\}$ where $v = 2\pi i \in \Lambda$, then $\mathcal{P} = [0, \infty)$ and $D_{-\mathcal{S}}$ is $\mathbb{C}$ with the standard symplectic structure and circle action. So $M_{\mathcal{P}}$ is equal to the symplectic cut $M_{\geq 0}$ and $M_{-\mathcal{P}}$ is the opposite cut $M_{\leq 0}$. If $\mathcal{P}' = \{0\}$ then $M_{\mathcal{P}'} = M_0$. Admissibility of $\mathcal{P}$ is equivalent to 0 being a regular value of Φ. If we multiply the labelling vector v by k, then

$\mathcal{P}$ does not change, but the orbifold atlas on $M_{\mathcal{P}}$ changes and $M_{\mathcal{P}'}$ becomes the symplectic quotient M_0 counted k times.

It is useful to rephrase (5.9) in combinatorial terms. Consider the infinitesimal orbit type stratification

$$M = \bigcup_{\beta \in \mathfrak{B}} M_\beta \tag{5.10}$$

of M and denote the subalgebra corresponding to β by $\mathfrak{t}_\beta$. Choose a basepoint $m_\beta \in M_\beta$ and put $\mu_\beta = \Phi(m_\beta)$ for all β. The closure of M_β is a T-invariant symplectic suborbifold of M; the affine span of the convex polytope $\Phi(\bar{M}_\beta)$ is $\mu_\beta + \mathfrak{t}_\beta^0$ and its relative interior is $\Phi(M_\beta)$. (See e. g. [17].) The sets $\Phi(M_\beta)$ are called the *virtual open faces* of Δ. Since $\mathfrak{t}_{\mathcal{F}}$ is the annihilator of the tangent space of $\mathcal{F}$, the following statement is clear.

Lemma 5.4. *A polyhedron $\mathcal{P}$ defined by a set of labels of constant excess is admissible if and only if its open faces are transverse to the virtual open faces of Δ. Consequently, admissibility is a generic condition.*

If $\mathcal{P}$ is admissible, the pullback $\mathrm{pr}_M^* L$ of L to $M \times D_{-\mathcal{S}}$ is almost equivariantly locally trivial at level 0, and so the *cut bundle*

$$L_{\mathcal{P}} = \mathrm{pr}_M^* L /\!\!/ T$$

is a well-defined line orbibundle on $M_{\mathcal{P}}$. Likewise, L induces well-defined orbibundles $L_{\mathcal{P}'}$ on each of the cuts $M_{\mathcal{P}'}$. From (3.18) we obtain for every open face $\mathcal{F}$ of $\mathcal{P}$ a canonical isomorphism

$$L_{\mathcal{P}}\big|_{\Phi_{\mathcal{P}}^{-1}(\mathcal{F})/T_{\mathcal{F}}} \cong \big(L|_{\Phi^{-1}(\mathcal{F})}\big)/T_{\mathcal{F}}. \tag{5.11}$$

Definition 5.5. An *admissible* or *T-admissible polyhedral subdivision* of $\mathfrak{t}^*$ is a collection $\mathfrak{P}$ satisfying the following conditions: every element of $\mathfrak{P}$ is a T-admissible polyhedron in $\mathfrak{t}^*$, their union is $\mathfrak{t}^*$, for every element of $\mathfrak{P}$ all its closed faces are in $\mathfrak{P}$, and the intersection of any two elements of $\mathfrak{P}$ is a closed face of each.

Theorem 5.6 (gluing formula, [34])**.** *Let $\mathfrak{P}$ be an admissible polyhedral subdivision of $\mathfrak{t}^*$. Then*

$$\mathrm{RR}(M, L) = \sum_{\mathcal{P} \in \mathfrak{P}} (-1)^{\operatorname{codim} \mathcal{P}} \, \mathrm{RR}(M_{\mathcal{P}}, L_{\mathcal{P}}) \tag{5.12}$$

as virtual characters of T. If $T = S^1$ and the cutting data are only locally defined, then we have a numerical identity

$$\mathrm{RR}(M, L) = \mathrm{RR}(M_{\geq 0}, L_{\geq 0}) + \mathrm{RR}(M_{\leq 0}, L_{\leq 0}) - \mathrm{RR}(M_0, L_0). \tag{5.13}$$

In the presence of a compact connected Lie group H that acts on M and L in such a way that H commutes with T and the H-action on M is symplectic, the equalities (5.12) *and* (5.13) *hold as identities of virtual characters of H.* □

The orbifold structures of $M_{\mathcal{P}}$ and $L_{\mathcal{P}}$ depend not only on $\mathcal{P}$, but also on the sets of labels defining them; cf. Example 5.3. However, by Proposition 4.4 of [34] the equivariant character $\mathrm{RR}(M_{\mathcal{P}}, L_{\mathcal{P}})$ depends only on the underlying polyhedron, so (5.12) and (5.13) make sense.

5.2.2. *Multiplicities.* If the origin is a vertex of the weight polytope Δ_L, then the right-hand side of (5.1) has at most one nonzero summand, which corresponds to a certain vertex of the moment polytope Δ.

Proposition 5.7. *Suppose that* 0 *is a vertex of the weight polytope* Δ_L*. Choose a generic* $\xi \in \mathfrak{t}$ *with the property that* $\langle \sigma_F, \xi \rangle > 0$ *for all* F *with* $\sigma_F \neq 0$*. Let* ν *be the vertex of the moment polytope* Δ *where the function sending* ρ *to* $\langle \rho, \xi \rangle$ *attains its minimum. Then*

$$\mathrm{RR}(M,L)^T = \begin{cases} \mathrm{RR}\big(\Phi^{-1}(\nu), L|_{\Phi^{-1}(\nu)}\big) = \mathrm{RR}(M_\nu, L_\nu) & \text{if } \sigma_{\Phi^{-1}(\nu)} = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{5.14}$$

If L *is rigid, then* $\mathrm{RR}(M,L) = \mathrm{RR}(M_\mu, L_\mu)$ *for all vertices* μ *of* Δ*. If* L *is a moment bundle, then* $\mathrm{RR}(M,L)^T = \mathrm{RR}(M_0, L_0)$ *and*

$$\mathrm{RR}\big(M, L^{-1}\big)^T = \begin{cases} \mathrm{RR}\big(M_0, L_0^{-1}\big) & \text{if } \Delta = \{0\}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Note first that for every fixed-point component F the cone $\mathcal{C}_F$ is equal up to a translation to the cone with vertex Φ_F spanned by Δ:

$$\mathcal{C}_F = -\Phi(F) + \mathrm{cone}_{\Phi(F)}(\Delta). \tag{5.15}$$

In particular, $\mathcal{C}_F = \mathfrak{t}^*$ for $\Phi(F)$ in the interior of Δ. Secondly, if μ is any vertex of Δ, then $\Phi^{-1}(\mu)$ is a connected component of M^T, and so $M_\mu = \Phi^{-1}(\mu)$. Further, if ξ is generic, then $\rho \mapsto \langle \rho, \xi \rangle$ attains its minimum at a unique vertex of Δ, so ν is well-defined. If $\sigma_{\Phi^{-1}(\nu)} = 0$, then T acts trivially on $L|_{\Phi^{-1}(\nu)}$ and so $L_\mu = L|_{\Phi^{-1}(\nu)}$.

If F is a fixed-point component for which $\xi \in \check{\mathcal{C}}_F$, then by (5.15) the moment polytope is contained in the halfspace given by $\langle \rho, \xi \rangle \geq \langle \Phi(F), \xi \rangle$. In other words, $\rho \mapsto \langle \rho, \xi \rangle$ attains its minimum at $\Phi(F)$, so $\Phi(F) = \nu$. The equality (5.14) now follows immediately from Theorem 5.1 by setting $\mu = 0$.

If L is rigid, then $\Delta_L = \{0\}$, so $\sigma_{\Phi^{-1}(\nu)} = 0$. Moreover, $\mathrm{RR}(M,L)$ is a constant character by 1 of Theorem 2.7, so (5.14) implies $\mathrm{RR}(M,L) = \mathrm{RR}(M,L)^T = \mathrm{RR}(M_\nu, L_\nu)$. By varying the choice of ξ we obtain this equality for all vertices ν. If L is a moment bundle, then $\Delta_L = \Delta$, so $\nu = 0$. Hence $\mathrm{RR}(M,L)^T = \mathrm{RR}(M_0, L_0)$ by (5.14). The weight polytope of the dual moment bundle L^{-1} is $\Delta_{L^{-1}} = -\Delta$. So if $\Delta = \{0\}$, then $\nu = 0$ and $\mathrm{RR}\big(M, L^{-1}\big)^T = \mathrm{RR}\big(M_0, L_0^{-1}\big)$, but if $\Delta \neq \{0\}$, then ν is a vertex distinct from 0, in which case $\sigma_{\Phi^{-1}(\nu)} = -\nu \neq 0$, so $\mathrm{RR}\big(M, L^{-1}\big)^T = 0$ by (5.14). ∎

An immediate consequence of this result and the gluing formula is the invariance of the index under blowing up. Let S be a closed symplectic suborbifold of M and suppose that S^1 acts on an open neighbourhood U of S with fixed-point set S and with positive weights α. Let $\mathrm{Bl}(M, S, \varepsilon, \alpha)$ be a weighted blowup of M at S as defined at the end of Section 4.1.1 and let $\mathrm{Bl}(L, S, \varepsilon, \alpha)$ be the weighted blowup bundle.

Theorem 5.8.

$$\mathrm{RR}\big(\mathrm{Bl}(M, S, \varepsilon, \alpha), \mathrm{Bl}(L, S, \varepsilon, \alpha)\big) = \mathrm{RR}(M, L). \tag{5.16}$$

Proof. The blowup is by definition the symplectic cut $M_{\leq 0}$ with respect to the function $\psi + \varepsilon$, where ψ is the function generating the circle action on U. The

exceptional divisor M_0 is the weighted projectivization $\mathbb{P}_\alpha N$, where N is the symplectic normal bundle of S, and $M_{\geq 0}$ is the weighted projectivization $\mathbb{P}_\alpha(N \oplus \mathbb{C})$. By Definition 4.3, $L_{\geq 0}$ is S^1-rigid with respect to the residual S^1-action on $M_{\geq 0}$. The minimum of the moment function on $M_{\geq 0}$ is 0 and the fibre over 0 is M_0. Hence $\mathrm{RR}(M_{\geq 0}, L_{\geq 0}) = \mathrm{RR}(M_0, L_0)$ by Proposition 5.7 and therefore $\mathrm{RR}(M, L) = \mathrm{RR}(M_{\leq 0}, L_{\leq 0})$ by (5.13). □

Let H be a compact connected Lie group that acts on M and L in such a way that H commutes with T and the H-action on M is symplectic. The following assertion is evident from 3 of Theorem 5.1.

Addendum 5.9. *The equalities* (5.14) *and* (5.16) *hold as identities of virtual characters of H.* □

To put (5.12) and Proposition 5.7 together we need to calculate, for any admissible polytope $\mathcal{P}$, the fixed points of the T-action on $M_{\mathcal{P}}$ and the weights of the T-action on the fibres of $L_{\mathcal{P}}$ at $M_{\mathcal{P}}^T$. Let m be in the stratum M_β defined by (5.10), so that $\mathfrak{t}_m = \mathfrak{t}_\beta$, and suppose that $\Phi(m)$ lies in an open face $\mathcal{F}$ of $\mathcal{P}$. Then by (5.8) the infinitesimal stabilizer of the image of m in $M_{\mathcal{P}}$ is equal to $\mathfrak{t}_\beta + \mathfrak{t}_{\mathcal{F}}$. By (5.9) this sum is direct. Therefore, the connected components of $M_{\mathcal{P}}^T$ are the orbifolds

$$\big(M_\beta \cap \Phi^{-1}(\mathcal{F})\big)/T_{\mathcal{F}} \tag{5.17}$$

for all $\beta \in \mathfrak{B}$ and all open faces $\mathcal{F}$ of $\mathcal{P}$ such that $\mathfrak{t}_\beta \oplus \mathfrak{t}_{\mathcal{F}} = \mathfrak{t}$. (These sets are connected, because $M_\beta \cap \Phi^{-1}(\mathcal{F})$ is exactly the open stratum of a fibre of the $T_{\mathcal{F}}$-moment map on $\bar{M}_\beta$.) Let $\sigma_\beta \in \mathfrak{t}_\beta^*$ denote the orbiweight of the $\mathfrak{t}_\beta$-action on $L|_{M_\beta}$ and $\bar{\sigma}_{\beta\mathcal{F}}$ the orbiweight of the $\mathfrak{t}$-action on the restriction of $L_{\mathcal{P}}$ to the fixed-point component (5.17). Then (5.11) implies

$$\bar{\sigma}_{\beta\mathcal{F}} = (\sigma_\beta, 0) \in \mathfrak{t}_\beta^* \oplus \mathfrak{t}_{\mathcal{F}}^* = \mathfrak{t}^*, \tag{5.18}$$

which proves the first part of the following lemma.

Lemma 5.10. 1. *The orbiweight $\bar{\sigma}_{\beta\mathcal{F}}$ is the projection of σ_β onto the tangent space of $\mathcal{F}$ along the affine space spanned by the virtual face $\Phi(M_\beta)$.*

2. *If L is almost equivariantly locally trivial at level μ, then there exists a neighbourhood O of μ in $\mathfrak{t}^*$ such that for every admissible polyhedron $\mathcal{P} \subset O$ the cut bundle $L_{\mathcal{P}}$ is rigid on $M_{\mathcal{P}}$.*
3. *If L is rigid, then so is $L_{\mathcal{P}}$ for any admissible polyhedron $\mathcal{P}$.*
4. *Let $\mathcal{P}$ be an admissible polyhedron and assume L is a moment bundle. Then $L_{\mathcal{P}}$ is a moment bundle on $M_{\mathcal{P}}$ if and only if the affine subspace spanned by $\mathcal{F}$ contains the origin for every open face $\mathcal{F}$ of $\mathcal{P}$ such that $\Delta \cap \mathcal{F}$ is nonempty.*
5. *If L is a moment bundle and $\mathcal{P}$ is an admissible cone with apex at the origin, then $L_{\mathcal{P}}$ is a moment bundle. If $\mathcal{P}'$ is the shifted cone $\mu + \mathcal{P}$, where $\mu \in \mathfrak{t}^*$, then the weight polytope of $L_{\mathcal{P}'}$ is contained in $\mathcal{P}$ when μ is sufficiently small. Hence, by Theorem 5.1, the support of the multiplicity function of $L_{\mathcal{P}'}$ is contained in $\mathcal{P}$.*

Proof. If L is almost equivariantly locally trivial at level μ, then by Lemma 3.7 there exists a neighbourhood O of μ in $\mathfrak{t}^*$ such that L is almost equivariantly locally trivial on $\Phi^{-1}(O)$. This implies that if $\mathcal{P}$ lies in O, then $\sigma_\beta = 0$ for all β and $\mathcal{F}$ such that $M_\beta \cap \Phi^{-1}(\mathcal{F})$ is nonempty. Therefore $L_{\mathcal{P}}$ is rigid by 1. This proves 2.

If L is rigid, then $\sigma_\beta = 0$ for all β by Lemma 3.11 and hence $L_{\mathcal{P}}$ is rigid for any admissible $\mathcal{P}$ by 1. This proves 3.

For the proof of 4, let $\iota_{\mathcal{F}}$ and ι_β denote the inclusions of $\mathfrak{t}_{\mathcal{F}}$, resp. $\mathfrak{t}_\beta$, into $\mathfrak{t}$. Since L is a moment bundle, $\sigma_\beta = \iota_\beta^*\Phi(m)$ by Lemma 3.11. The cut bundle $L_{\mathcal{P}}$ is therefore a moment bundle if and only if $\sigma_\beta = \Phi(m) = \bigl(\iota_{\mathcal{F}}^*\Phi(m), \iota_\beta^*\Phi(m)\bigr)$ for all open faces $\mathcal{F}$ of $\mathcal{P}$ and all $m \in \Phi^{-1}(\mathcal{F})$ such that $\mathfrak{t}_m = \mathfrak{t}_\beta$. This condition amounts to $\iota_{\mathcal{F}}^*\Phi(m) = 0$ for all $m \in \Phi^{-1}(\mathcal{F})$, which is equivalent to $\mathcal{F} \subset (\mathfrak{t}_{\mathcal{F}})^0$ whenever $\Phi(M) \cap \mathcal{F}$ is nonempty.

The first statement under 5 is evident from 4. For the second statement consider an arbitrary orbiweight $\bar{\sigma}_{\beta\mathcal{F}'}$ of $L_{\mathcal{P}'}$. By 1 it is equal to the projection of $\Phi(m)$ onto the tangent space of $\mathcal{F}'$ for some $m \in M_\beta$. Here the open face $\mathcal{F}'$ of $\mathcal{P}'$ and $\beta \in \mathfrak{B}$ are such that $\mathcal{F}'$ and $\Phi(M_\beta)$ intersect transversely at $\Phi(m)$. Now $\mathcal{F}'$ is of the form $\mu + \mathcal{F}$, where $\mathcal{F}$ is an open face of $\mathcal{P}$. If $\mathcal{F} = \{0\}$ then $\bar{\sigma}_{\beta\mathcal{F}'} = 0$, which is in $\mathcal{P}$. If $\mathcal{F} \neq \{0\}$, let us choose an inner product on $\mathfrak{t}^*$ such that the decomposition $\mathfrak{t}^* = \mathfrak{t}_\beta^* \oplus \mathfrak{t}_{\mathcal{F}}^*$ is orthogonal. Then the distance d of $\Phi(m) \in \mathcal{F}'$ to the boundary of $\mathcal{F}'$ is positive. This implies that as long as $|\mu| < d$ the projection of $\Phi(m)$ onto $\mathcal{F}$ is contained in the interior of $\mathcal{F}$, so in particular $\bar{\sigma}_{\beta\mathcal{F}'}$ is in $\mathcal{P}$. Since the number of orbiweights $\bar{\sigma}_{\beta\mathcal{F}'}$ is finite, we get only finitely many such conditions on μ and conclude that $\Delta_{L_{\mathcal{P}'}} \subset \mathcal{P}$ if μ is sufficiently small. ☐

Proof of Theorem 2.7 (*abelian case*). The proof of 1 is in Section 5.1. For the proof of 2 we may assume that T acts effectively on M and, after shifting the moment map if necessary, that $\mu = 0$. Let Δ' be the unique closed face of Δ that contains 0 in its (relative) interior. Then Δ' is the image under Φ of a component M' of the fixed-point set of a certain subtorus T' of T. Now $\mathrm{pr}_{(\mathfrak{t}')^*}(\Delta')$ is a vertex of $\mathrm{pr}_{(\mathfrak{t}')^*}(\Delta)$, which is the moment polytope of M for the T'-action, and the symplectic quotient of M at $\mathrm{pr}_{(\mathfrak{t}')^*}(\Delta')$ with respect to T' is M'. Proposition 5.7 implies that $\mathrm{RR}(M, L) = \mathrm{RR}(M', L|_{M'})$. We may therefore assume that 0 is in the interior of Δ.

Consider first the case that 0 is a quasi-regular value of Φ. Then 0 is in fact a regular value, because every fibre over an interior point of Δ intersects the principal stratum of M and T acts generically freely. This implies that there exists an admissible polyhedral subdivision $\mathfrak{P}$ of $\mathfrak{t}^*$ such that $0 \in \mathfrak{P}$. (For instance, we can take each element of $\mathfrak{P}$ to be a suitable simplicial cone with vertex 0.) Then $L_{\mathcal{P}}$ is rigid for every $\mathcal{P} \in \mathfrak{P}$ by Lemma 5.10 and 0 is a vertex of every $\mathcal{P}$. Consequently $\mathrm{RR}(M_{\mathcal{P}}, L_{\mathcal{P}}) = \mathrm{RR}(M_0, L_0)$ by Proposition 5.7. Combining the Euler identity

$$\sum_{\mathcal{P} \in \mathfrak{P}} (-1)^{\operatorname{codim} \mathcal{P}} = 1 \tag{5.19}$$

with (5.12) we infer that $\mathrm{RR}(M, L) = \mathrm{RR}(M_0, L_0)$.

Consider finally the case that 0 is a singular value of Φ. Since T is abelian and M is compact, the blowups at each stage of the desingularization process of Section 4.1.2 are globally defined and give rise to a compact Hamiltonian T-orbifold $\bigl(\tilde{M}, \tilde{\omega}, \tilde{\Phi}\bigr)$ and a globally defined rigid orbibundle $\tilde{L}$ on $\tilde{M}$. Theorem 5.8 implies that $\mathrm{RR}(M, L) = \mathrm{RR}\bigl(\tilde{M}, \tilde{L}\bigr)$. Because 0 is a quasi-regular value of $\tilde{\Phi}$, the T-equivariant part of the latter is equal to $\mathrm{RR}\bigl(\tilde{M}_0, \tilde{L}_0\bigr)$, which is equal to $\mathrm{RR}(M_0, L_0)$ by Definition 2.3. ☐

Proof of Theorem 2.5 (*abelian case*). By Lemma 5.10 there exists a neighbourhood O of μ such that $L_{\mathcal{P}}$ is rigid on $M_{\mathcal{P}}$ for all admissible $\mathcal{P} \subset O$. Let ν be any point in O. Choose any set of labels $\mathcal{S} = \bigl\{(v_1, r_1), (v_2, r_2), \dots, (v_n, r_n)\bigr\}$ such that the

associated polyhedron $\mathcal{P}$ has dimension $\dim T$, is contained in O, and contains μ and ν in its interior. By Lemma 5.4 admissibility is a generic condition, so a small perturbation of the parameters $(r_1, \dots, r_n)$ will change $\mathcal{P}$ into an admissible polytope that still satisfies $\mu,\ \nu \in \mathcal{P}$ and $\mathcal{P} \subset O$. Then $L_{\mathcal{P}}$ is rigid on $M_{\mathcal{P}}$, so Theorem 2.7 implies $\mathrm{RR}(M_\mu, L_\mu) = \mathrm{RR}(M_{\mathcal{P}}, L_{\mathcal{P}}) = \mathrm{RR}(M_\nu, L_\nu)$. □

Proof of Theorem 2.9 (*abelian case*). As in the proof of Theorem 2.7 we can reduce the general case to the case where T acts effectively and that 0 is an interior point of Δ. (Here we apply Proposition 5.7 and Addendum 5.9, where $H = T/T'$.)

Again, we handle first the case that 0 is a regular value of Φ. Let $\mathfrak{P}$ be an admissible polyhedral subdivision of $\mathfrak{t}^*$ consisting of cones centred at the origin. Then $L_{\mathcal{P}}$ is a moment bundle on $M_{\mathcal{P}}$ for every $\mathcal{P} \in \mathfrak{P}$ by Lemma 5.10. By Proposition 5.7,

$$\mathrm{RR}(M_{\mathcal{P}}, L_{\mathcal{P}})^T = \mathrm{RR}(M_0, L_0), \tag{5.20}$$

because 0 is a vertex of $\Delta \cap \mathcal{P}$. Applying the gluing formula and (5.19) we find $\mathrm{RR}(M, L)^T = \mathrm{RR}(M_0, L_0)$.

If 0 is a singular value, we can choose $\mathfrak{P}$ such that the shifted cones $\mathcal{P}' = \mu + \mathcal{P}$ for $\mathcal{P} \in \mathfrak{P}$ are admissible for μ sufficiently close to 0. The weight for the T-action on $L_\mu = L_{\mathcal{P}'}|_{\Phi_{\mathcal{P}'}^{-1}(\mu)}$ is trivial and the set of weights for the T-action on $L_{\mathcal{P}'}|_{M_{\mathcal{P}'}^T}$ is contained in $\mathcal{P}$ by Lemma 5.10. Hence

$$\mathrm{RR}(M_{\mathcal{P}'}, L_{\mathcal{P}'})^T = \mathrm{RR}(M_\mu, L_\mu) \tag{5.21}$$

by Proposition 5.7. Putting together the gluing formula, (5.19) and Theorem 2.5 we obtain $\mathrm{RR}(M, L)^T = \mathrm{RR}(M_0, L_0)$. □

Proof of Theorem 2.14 (*abelian case*). Assuming that the theorem is true, by taking T-invariants on both sides we deduce that

$$N_{L^{-1}}(0) = \begin{cases} (-1)^{\dim \Delta}\, \mathrm{RR}\big(M_0, L_0^{-1}\big) & \text{if } 0 \in \operatorname{int} \Delta, \\ 0 & \text{otherwise.} \end{cases} \tag{5.22}$$

We assert that the theorem is in fact equivalent to (5.22). This follows from a variant of the shifting trick, which in this abelian situation allows us to write

$$\begin{aligned} N_{L^{-1}}(\mu) &= \mathrm{RR}\big(M \times \{\mu\}, L^{-1} \boxtimes (E_{-\mu})^{-1}\big)^T = \mathrm{RR}\big(M \times \{\mu\}, (L \boxtimes E_{-\mu})^{-1}\big)^T \\ &= \begin{cases} (-1)^{\dim \Delta}\, \mathrm{RR}\big(M_{-\mu}, (L_{-\mu}^{\mathrm{shift}})^{-1}\big) & \text{if } -\mu \in \operatorname{int} \Delta, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{5.23}$$

The first equality follows from the Künneth formula and the fact that the character $\zeta_{-\mu}$ is dual to ζ_μ; and the third equality follows from (5.22). It is clear that (5.23) implies Theorem 2.14.

The proof of (5.22) proceeds in the same way as the proof of Theorem 2.9. The only difference is that (5.20) is replaced by

$$\mathrm{RR}\big(M_{\mathcal{P}}, L_{\mathcal{P}}^{-1}\big)^T = \begin{cases} \mathrm{RR}\big(M_0, L_0^{-1}\big) & \text{if } \mathcal{P} = \{0\}, \\ 0 & \text{otherwise,} \end{cases}$$

and (5.21) by

$$\mathrm{RR}\big(M_{\mathcal{P}'}, L_{\mathcal{P}'}^{-1}\big)^T = \begin{cases} \mathrm{RR}\big(M_\mu, L_\mu^{-1}\big) & \text{if } \mathcal{P} = \{0\}, \\ 0 & \text{otherwise,} \end{cases}$$

both of which follow from Proposition 5.7. □

5.2.3. *Asymptotic moment bundles.* The above proofs of Theorems 2.9 and 2.14 do not generalize directly to the nonabelian case. We need to extend the discussion to a class of orbibundles that are "almost" moment bundles. Let $\mathfrak{M}$ be a T-orbifold fibring over the interval $(0,1]$. Assume that the bundle projection has compact fibres and is T-invariant. Let $\mathfrak{L}$ be a line orbibundle over $\mathfrak{M}$ and (ω,Φ) a relative equivariant symplectic form on $\mathfrak{M}$ in the sense of Appendix A.2. Let us denote the fibre of $\mathfrak{M}$ over t by M^t and the restrictions of $\mathfrak{L}$, ω and Φ to M^t by L^t, ω^t and Φ^t, respectively. Then all fibres M^t are Hamiltonian T-orbifolds; they are equivariantly diffeomorphic (but not necessarily symplectomorphic) to one another; and the orbibundles L^t are equivariantly isomorphic to one another. It follows that the T-character $\mathrm{RR}\big(M^t,L^t\big)$ is independent of t.

We call $\mathfrak{L}$ an *asymptotic moment bundle* if for all components $\mathfrak{F}$ of the fixed-point set $\mathfrak{M}^T$ the limit of $\Phi(\mathfrak{F}\cap M^t)$ as $t\to 0$ exists and is equal to the orbiweight of the T-action on $\mathfrak{L}|_{\mathfrak{F}}$. Here $\Phi(\mathfrak{F}\cap M^t)$ is the (constant) value of Φ on $\mathfrak{F}\cap M^t$. Let Δ^t denote the moment polytope of M^t; then the limit polytope $\Delta=\lim_{t\to 0}\Delta^t$ is well-defined. As in Lemma 3.11 one shows that $\sigma_m=\lim_{t\to 0}\iota_m^*\Phi^t(m)$, where $\sigma\in\mathfrak{t}_m^*$ is the character defining the action of $(T_m)^0$ on L_m and $\iota_m\colon\mathfrak{t}_m\to\mathfrak{t}$ is the inclusion.

Let us now for each $\mu\in\Delta$ select a path $\gamma_\mu(t)$ in $\mathfrak{t}^*$ defined for $0<t\le 1$ such that $\gamma_\mu(t)\in\Delta^t$ and $\lim_{t\to 0}\gamma_\mu(t)=\mu$. Then for all $m\in M^t$ such that $\Phi(m)=\gamma_\mu(t)$ we have $\sigma_m=\lim_{t\to 0}\gamma_\mu(t)=\mu$. It follows that for all μ and all t the orbibundle $L^t\boxtimes E_{-\mu}$ is almost equivariantly locally trivial at level $\gamma_\mu(t)$, so that

$$(L^t)^{\mathrm{shift}}_{\gamma_\mu(t)}=(L^t\boxtimes E_{-\mu})\big|_{(\Phi^t)^{-1}(\gamma_\mu(t))}\Big/T$$

is a well-defined orbibundle on $M^t_{\gamma_\mu(t)}$. The proof of the following result is completely analogous to that of Theorems 2.9 and 2.14 in the abelian case.

Theorem 5.11. *Let $\mathfrak{L}$ be an asymptotic moment bundle on $\mathfrak{M}$. Then for $0<t\le 1$*

$$\mathrm{RR}(M^t,L^t)=\sum_{\mu\in\Lambda^*\cap\Delta}\mathrm{RR}\big(M^t_{\gamma_\mu(t)},(L^t)^{\mathrm{shift}}_{\gamma_\mu(t)}\big)\,\zeta_\mu\qquad\textit{and}$$

$$\mathrm{RR}\big(M^t,(L^t)^{-1}\big)=(-1)^{\dim\Delta}\sum_{\mu\in\Lambda^*\cap\operatorname{int}\Delta}\mathrm{RR}\Big(M^t_{\gamma_\mu(t)},\big((L^t)^{\mathrm{shift}}_{\gamma_\mu(t)}\big)^{-1}\Big)\,\zeta_{-\mu}.\quad\square$$

5.3. **Delzant spaces III.** Let $\mathcal{S}=\big\{(v_1,r_1),(v_2,r_2),\dots,(v_n,r_n)\big\}$ be a set of labels for the torus T. Assume that the associated polyhedron $\mathcal{P}$ is nonempty and compact and that its dimension is equal to $k=\dim T$. Suppose that the r_i are integers, so that $\mathcal{P}$ is a rational polyhedron. We assert that the Delzant space $D_{\mathcal{S}}$ is prequantizable in the sense of Example 2.8.

Indeed, the vector $r=(r_1,\dots,r_n)\in(\mathbb{Z}^n)^*$ defines a real infinitesimal weight of the torus $T^n=\mathbb{R}^n/\mathbb{Z}^n$. This implies that the cotangent bundle T^*T is $T^n\times T$-equivariantly prequantizable. The prequantum line bundle is the trivial line bundle $L_{T^*T}=T^*T\times\mathbb{C}$, where T^n acts with weight $-r$ on the fibre and T acts trivially on the fibre. By (2.3) the T^n-moment map corresponding to this equivariant line bundle is the map ψ_r given by (3.20). A $T^n\times T$-equivariant prequantum line bundle on $\mathbb{C}^n$ is the trivial line bundle $L_{\mathbb{C}^n}=\mathbb{C}^n\times\mathbb{C}$, where T^n and T both act trivially on the fibre. The bundle $L=L_{T^*T}\boxtimes L_{\mathbb{C}^n}$ is then a $T^n\times T$-equivariant prequantum line bundle on $T^*T\times\mathbb{C}^n$ and the associated moment map for the T^n-action is given by

(3.21). The upshot is that the quotient $L_{\mathcal{S}} = L /\!\!/ T^n$ is a T-equivariant prequantum line orbibundle on the Delzant space $D_{\mathcal{S}}$ and that the associated moment map is $\Psi_{\mathcal{S}}$.

If $\mathcal{P}$ is a lattice polytope (i. e. all its vertices are in Λ^*), then it is not difficult to see that L_{T^*T} is T^n-equivariantly locally trivial, so L_0 is in fact a genuine line bundle.

By Proposition 3.15, for $m \in \mathbb{Z}$ the Delzant space $D_{m\mathcal{S}}$ is symplectomorphic to $D_{\mathcal{S}}$ with m times its symplectic form. It is not hard to check that under this symplectomorphism $L_{m\mathcal{S}}$ pulls back to the mth tensor power $L_{\mathcal{S}}^m$. The next result follows immediately from Theorems 2.9 and 2.14 and the fact that $D_{\mathcal{S}}$ is multiplicity-free.

Proposition 5.12. *For all $m \in \mathbb{N}$*

$$\operatorname{RR}(D_{\mathcal{S}}, L_{\mathcal{S}}^m)(z) = \sum_{\mu \in \Lambda^* \cap m\mathcal{P}} z^\mu \qquad \text{and}$$
$$\operatorname{RR}(D_{\mathcal{S}}, L_{\mathcal{S}}^{-m})(z) = (-1)^k \sum_{\mu \in \Lambda^* \cap \operatorname{int}(m\mathcal{P})} z^{-\mu}$$

as rational functions on $T^{\mathbb{C}}$. □

For every nonnegative integer m, let $p(m)$ denote the number of lattice points in $m\mathcal{P}$. Then by Proposition 5.12, $\operatorname{RR}(D_{\mathcal{S}}, L_{\mathcal{S}}^m)(1) = p(m)$ and $\operatorname{RR}(D_{\mathcal{S}}, L_{\mathcal{S}}^{-m})(1)$ is $(-1)^k$ times the number of lattice points in the interior of $m\mathcal{P}$. The following result was proved for simple rational polytopes in [35].

Corollary 5.13. *The counting function p is a quasi-polynomial, whose period is a divisor of the smallest positive integer l such that $l\mathcal{P}$ is a lattice polytope. It satisfies the Ehrhart reciprocity law $p(-m) = (-1)^k \#\big(\Lambda^* \cap \operatorname{int}(mP)\big)$.*

Proof. We defined $D_{\mathcal{S}}$ as the symplectic quotient M_0 of $M = T^*T \times \mathbb{C}^n$ and $L_{\mathcal{S}}$ as the quotient L_0 of the line bundle $L = L_{T^*T} \boxtimes L_{\mathbb{C}^n}$. Select a small regular value η of $\tilde{\psi}_r$; then $\operatorname{RR}(D_{\mathcal{S}}, L_{\mathcal{S}}^m) = \operatorname{RR}(M_\eta, L_\eta^m)$ for all $m \in \mathbb{Z}$ by Theorem 2.5. The result now follows from Proposition 5.12 and the Kawasaki-Riemann-Roch formula for M_η and L_η^m. □

This implies the well-known result that the counting function of a lattice polytope is polynomial. (See e. g. [14].) Recall from Section 4.4 that the shift desingularization M_η is none other than the Delzant space associated to the labelled polytope $(\mathcal{S}_\eta, \mathcal{P}_\eta)$, where $\mathcal{S}_\eta$ is as in (4.2). Guillemin pointed out in [16] that Proposition 5.12 leads to an Euler-MacLaurin type formula for the number of lattice points in $\mathcal{P}$, namely

$$p(1) = \lim_{\eta \to 0} \sum_{\mathcal{F} \preccurlyeq \mathcal{P}} \mathfrak{T}_{\mathcal{F}}\left(\frac{\partial}{\partial \eta}\right) \operatorname{vol} \mathcal{P}_\eta. \tag{5.24}$$

Here $\operatorname{vol} \mathcal{P}_\eta$ is the normalized Euclidean volume of $\mathcal{P}_\eta$ and the $\mathfrak{T}_{\mathcal{F}}$ are certain infinite-order differential operators associated to the faces of $\mathcal{P}$. These operators depend on the component of the set of regular values containing η; see [16] for details. There is an analogous formula involving the moment polytope $\tilde{\mathcal{P}}_\varepsilon$ associated to the canonical desingularization of $D_{\mathcal{S}}$. These identities are generalizations of the Khovanskii-Pukhlikov formula for the number of lattice points in a simply laced lattice polytope. Purely combinatorial proofs were given independently by Brion

and Vergne [7]. See their paper for a discussion of the relationship between (5.24) and similar formulæ proven by Cappell and Shaneson [9].

6. The general case

This section contains the proofs of the remaining theorems of Section 2 for general compact groups. We reduce the general case to the abelian case by means of the cross-section theorem and local symplectic cutting with respect to certain subtori of the maximal torus. In this section (M, ω, Φ) denotes a compact connected Hamiltonian G-orbifold with moment polytope Δ and L denotes a G-equivariant line orbibundle on M.

6.1. **Induction and cutting.** In this section we prove Theorems 2.5 and 2.7. We start by observing that Theorem 2.7 is true at "maximal" values of Φ. Let $\Phi_T = \mathrm{pr}_{\mathfrak{t}^*} \circ \Phi$ be the moment map for the T-action on M. Choose an invariant inner product on $\mathfrak{g}^*$ and let $|\cdot|$ denote the associated norm.

Lemma 6.1. *Let μ be a point in Δ of maximal norm. Then $\Phi^{-1}(\mu)$ is a component of the fixed-point set M^{G_μ} and is equal to $\Phi_T^{-1}(\mu)$. It follows that μ is a quasi-regular value of Φ.*

Proof. Recall that the moment polytope for the T-action is equal to $\Phi_T(M) = \mathrm{hull}\, \Phi(M^T) = \mathrm{hull}(\mathfrak{W} \cdot \Delta)$. This implies that μ is a vertex of $\Phi_T(M)$ and therefore $\Phi_T^{-1}(\mu) = \Phi^{-1}(\mu + \mathfrak{t}^0)$ is a component of M^T. Because the norm on $\mathfrak{g}^*$ is invariant we have $|\Phi(x)| \leq |\mu|$ for all $x \in M$, which for $x \in \Phi^{-1}(\mu + \mathfrak{t}^0)$ is only possible if $\Phi(x) = \mu$. This shows $\Phi^{-1}(\mu) = \Phi_T^{-1}(\mu)$. Now G_μ is a connected subgroup of maximal rank of G, so $\Phi^{-1}(\mu)$, being both invariant under the G_μ-action and fixed under the T-action, is fixed under G_μ. Since all points in $\Phi^{-1}(\mu)$ are of the same G_μ-orbit type, μ is a quasi-regular value. ☐

Putting this together with Theorem 2.7 (part 1 of which we proved in Section 5.1 and part 2 of which we proved in the abelian case in Section 5.2) and Addendum 5.9, we obtain the following.

Lemma 6.2. *Let μ be as in Lemma 6.1 and suppose L is rigid. Then $\mathrm{RR}(M, L) = \mathrm{RR}(M, L)^G = \mathrm{RR}(M_\mu, L_\mu)$. Given another Lie group H that acts on L and M in such a way that the action commutes with that of G and the action on M is symplectic, this holds as an equality of virtual characters of H.* ☐

What is the relationship between the equivariant index of M and that of its cross-sections Y_σ? Recall that for every open wall σ of the Weyl chamber $\mathfrak{t}^*_+$ the induction map $\mathrm{Ind}_{G_\sigma}^G$ is defined as the unique homomorphism $f\colon \mathrm{Rep}\, G_\sigma \to \mathrm{Rep}\, G$ such that $\mathrm{Ind}_T^G = f \circ \mathrm{Ind}_T^{G_\sigma}$. More specifically, let $\mu \in \Lambda^*_{\sigma,+} \subset \Lambda^*$ be a dominant weight for G_σ and let $\chi_{\sigma,\mu} \in \mathrm{Rep}\, G_\sigma$ be the corresponding irreducible character. Then

$$\mathrm{Ind}_{G_\sigma}^G \chi_{\sigma,\mu} = \mathrm{Ind}_T^G \zeta_\mu,$$

which is also equal to $\mathrm{RR}(G\mu, E_\mu)$ by (2.5).

Let Y_σ be the cross-section of M over σ as defined in Section 3.2. We say that Y_σ is a *global* cross-section of M if $M = GY_\sigma$, or equivalently, Δ is a subset of the open star $\bigcup_{\tau \succcurlyeq \sigma} \tau$ of σ.

Proposition 6.3.
1. *Let $\mathcal{Y}_\sigma$ be a compact almost complex G_σ-orbifold and $\mathcal{L}_\sigma$ a G_σ-equivariant line orbibundle on $\mathcal{Y}_\sigma$. Let $\mathcal{M}$ be the almost complex G-orbifold $G \times^{G_\sigma} \mathcal{Y}_\sigma$ equipped with the G-equivariant orbibundle $\mathcal{L}$ induced by $\mathcal{L}_\sigma$. Then* $\mathrm{RR}(\mathcal{M}, \mathcal{L}) = \mathrm{Ind}_{G_\sigma}^G \mathrm{RR}(\mathcal{Y}_\sigma, \mathcal{L}_\sigma)$.
2. *If Y_σ is a global cross-section of M, then* $\mathrm{RR}(M, L) = \mathrm{Ind}_{G_\sigma}^G \mathrm{RR}(Y_\sigma, L_\sigma)$.

Proof. The proof of 1 is closely analogous to the proof of the quantum cross-section theorem of [34]. If Y_σ is a global cross-section, then $M = G \times^{G_\sigma} Y_\sigma$ by Theorem 3.5, so 2 is evident from 1. $\square$

A global cross-section Y_σ is *nontrivial* if $Y_\sigma \neq M$, that is to say $\sigma \neq \mathfrak{a}^*$. Usually M does not possess nontrivial global cross-sections, but even then we can obtain information on its equivariant index by dint of nonabelian symplectic cutting, which was invented by Woodward [45]. It is based on the fact that M is the union $\bigcup_{\sigma \preccurlyeq \mathfrak{t}^*_+} M_\sigma$ of G-invariant open subsets $M_\sigma = GY_\sigma$, each of which carries a Hamiltonian action of the torus A_σ which commutes with the action of G. This action is defined by identifing M_σ with $G \times^{G_\sigma} Y_\sigma$ as in the symplectic cross-section theorem and extending the natural A_σ-action on Y_σ to an action on M_σ which commutes with G. In other words, for a G_σ-orbit $[g, y]$ in $G \times^{G_\sigma} Y_\sigma$ and $t \in A_\sigma$ we put $t \cdot [g, y] = [g, ty]$. This is well-defined because A_σ commutes with G_σ. The restriction of L to M_σ acquires likewise a A_σ-action that commutes with G. The moment map for A_σ is the unique G-invariant extension of the A_σ-moment map on Y_σ and can be described as follows. Let

$$\Phi_+ \colon M \to \mathfrak{t}^*_+$$

be the composition of Φ with the quotient mapping q defined in (3.13). The moment map of the A_σ-action on Y_σ is equal to $\mathrm{pr}_\sigma \circ \Phi|_{Y_\sigma}$, where pr_σ is the canonical projection $\mathfrak{t}^* \to \mathfrak{a}^*_\sigma$. Now observe that $\mathrm{pr}_\sigma = \mathrm{pr}_\sigma \circ q$ on $\mathfrak{g}^*_\sigma \subset \mathfrak{g}^*$, and hence $\mathrm{pr}_\sigma \circ \Phi = \mathrm{pr}_\sigma \circ \Phi_+$ on Y_σ. The A_σ-moment map on M_σ is therefore equal to the G-invariant map $\mathrm{pr}_\sigma \circ \Phi_+|_{M_\sigma}$. This is a smooth map for all σ, even though Φ_+ is in general not smooth.

Now let $\mathcal{S}$ be a set of labels in $\mathfrak{t}^*$ and $\mathcal{P}$ its associated polyhedron.

Definition 6.4. The pair $(\mathcal{S}, \mathcal{P})$ is *admissible* or *G-admissible* with respect to M if $\mathcal{S}$ has constant excess and the following conditions hold for all open faces $\mathcal{F}$ of $\mathcal{P}$:

1. for all walls σ such that $\sigma \cap \mathcal{F} \cap \Delta$ is nonempty, $T_\mathcal{F}$ is a subtorus of A_σ;
2. the action of $T_\mathcal{F}$ on $\Phi^{-1}(\mathcal{F} \cap \mathfrak{t}^*_+)$ is locally free.

As in the abelian case, admissibility depends only on the polyhedron $\mathcal{P}$, not on $\mathcal{S}$, and condition 2 is satisfied generically. Condition 1 is tantamount to: for all σ such that $\sigma \cap \mathcal{F} \cap \Delta \neq \emptyset$, the tangent space to $\mathcal{F}$ contains the annihilator of $\mathfrak{a}_\sigma$ in $\mathfrak{t}^*$; in other words the orthogonal complement of σ (with respect to any invariant inner product) is contained in $\mathcal{F}$. It implies that every wall σ has an open neighbourhood O_σ inside $\operatorname{star} \sigma$ such that

$$\mathcal{P} \cap O_\sigma \cap \Delta = \mathrm{pr}_\sigma^{-1}(\mathcal{P}_\sigma) \cap O_\sigma \cap \Delta,$$

where $\mathcal{P}_\sigma = \mathcal{P} \cap \mathfrak{a}^*_\sigma$. The symplectic cut $(M'_\sigma)_{\mathcal{P}_\sigma}$ of the $G \times A_\sigma$-invariant open subset $M'_\sigma = \Phi^{-1}(GO_\sigma)$ of M_σ with respect to the polyhedron $\mathcal{P}_\sigma$ is then well-defined and condition 2 implies that it is an orbifold. For $\sigma \preccurlyeq \tau$ there is a natural symplectic embedding of a G-invariant open subset of $(M'_\sigma)_{\mathcal{P}_\sigma}$ into $(M_\tau)_{\mathcal{P}_\tau}$ and the result

of gluing the $(M'_\sigma)_{\mathcal{P}_\sigma}$ together along these embeddings is a compact Hamiltonian G-orbifold $(M_{\mathcal{P}}, \omega_{\mathcal{P}}, \Phi_{\mathcal{P}})$, the *symplectic cut* of M with respect to $\mathcal{P}$. Its moment polytope $\Delta_{\mathcal{P}}$ is equal to $\Delta \cap \mathcal{P}$. The bundles $(L|_{M'_\sigma})_{P_\sigma}$ are likewise well-defined and can be pasted together to a global G-equivariant *cut bundle* $L_{\mathcal{P}}$ on $M_{\mathcal{P}}$. See [34] for details. Put $\Phi_{\mathcal{P},+} = q \circ \Phi_{\mathcal{P}}$. By analogy with (5.8) and (5.11), for every open face $\mathcal{F}$ of $\mathcal{P}$ there are canonical isomorphisms

$$\Phi_{\mathcal{P},+}^{-1}(\mathcal{F}) \cong \Phi_+^{-1}(\mathcal{F})/T_{\mathcal{F}},$$
$$L_{\mathcal{P}}\big|_{\Phi_{\mathcal{P},+}^{-1}(\mathcal{F})} \cong \Big(L\big|_{\Phi_+^{-1}(\mathcal{F})}\Big)\Big/T_{\mathcal{F}}.$$

Proof of Theorem 2.7. The proof of 1 is in Section 5.1. For the proof of 2 we consider first the case that Δ is contained in the degenerate wall $\mathfrak{a}^*$ of $\mathfrak{t}_+^*$, where $\mathfrak{a}$ is the centre of $\mathfrak{g}$. Then M is in effect a Hamiltonian A-orbifold and the theorem reduces to the abelian case, which was covered in Section 5.2.2.

Now consider the case that Δ is not contained in $\mathfrak{a}^*$. Here the proof is by induction on the dimension of M. We may assume that the result holds for all compact connected groups H and all Hamiltonian H-orbifolds Q with $\dim Q < \dim M$. By Lemma 6.2, $\mathrm{RR}(M, L) = \mathrm{RR}(M, L)^G = \mathrm{RR}(M_\mu, L_\mu)$ if $\mu \in \Delta$ is of maximal norm. It therefore suffices to check that $\mathrm{RR}(M_\mu, L_\mu)$ is independent of $\mu \in \Delta$.

First we show that $\mathrm{RR}(M_\mu, L_\mu)$ is constant on the complement in Δ of $\mathfrak{a}^*$. Let μ and ν be in $\Delta - \mathfrak{a}^*$ and let σ be the largest open wall of $\mathfrak{t}_+^*$ such that μ, $\nu \in \operatorname{star}\sigma$. Then $\sigma \neq \mathfrak{a}^*$ and hence $\dim Y_\sigma < \dim M$. Choose an admissible polytope $\mathcal{P}$ such that $\mathcal{P} \cap \mathfrak{t}_+^*$ is a subset of the star of σ and μ and ν are in $\mathcal{P}$. Put $Y_{\sigma,\mathcal{P}} = (Y_\sigma)_{\mathcal{P}}$ and $L_{\sigma,\mathcal{P}} = (L|_{Y_\sigma})_{\mathcal{P}}$. By the induction hypothesis the function that sends λ to $\mathrm{RR}\big((Y_{\sigma,\mathcal{P}})_\lambda, (L_{\sigma,\mathcal{P}})_\lambda\big)$ is constant on $\mathcal{P} \cap \Delta$. Moreover, $M_\lambda = (Y_{\sigma,\mathcal{P}})_\lambda$ and $L_\lambda = (L_{\sigma,\mathcal{P}})_\lambda$ for all λ in $\mathcal{P} \cap \Delta$, so the conclusion is $\mathrm{RR}(M_\mu, L_\mu) = \mathrm{RR}(M_\nu, L_\nu)$.

It remains to show that $\mathrm{RR}(M_\nu, L_\nu) = \mathrm{RR}(M_\mu, L_\mu)$ where $\mu \in \Delta \cap \mathfrak{a}^*$ and ν is a point close to μ and contained in the principal face Δ_{gen} of Δ. Because $\mathfrak{a}$ is the centre of $\mathfrak{g}$, we can shift the moment map by μ and may therefore assume that $\mu = 0$.

Assume that 0 is a quasi-regular value of Φ, so that $\Phi^{-1}(0) = Z_\alpha$ for some $\alpha \in \mathfrak{A}$. By Proposition 3.9, M_ν is a symplectic fibre orbibundle over M_0 with general fibre $(F_\alpha)_\nu$ and L_ν is the pullback of L_0. Because ν is a generic value of Φ, it is a quasi-regular value and $(F_\alpha)_\nu$ is an orbifold. By Theorem B.1 (see Appendix B) we have $\mathrm{RR}(M_\nu, L_\nu) = \mathrm{RR}(M_0, L_0)\,\mathrm{RR}\big((F_\alpha)_\nu, \mathbb{C}\big)$. Here $F_\alpha = (T^*G \times W_\alpha/\Upsilon_\alpha)/\!\!/G_\alpha$, so $F_\alpha/\!\!/G = (W_\alpha/\Upsilon_\alpha)/\!\!/G_\alpha$ is a point by Lemma 3.2. Moreover, $\dim F_\alpha \le \dim M_\nu < \dim M$, so $\mathrm{RR}\big((F_\alpha)_\nu, \mathbb{C}\big) = \mathrm{RR}\big((F_\alpha)_0, \mathbb{C}\big) = 1$ by the induction hypothesis. The upshot is $\mathrm{RR}(M_\nu, L_\nu) = \mathrm{RR}(M_0, L_0)$.

If 0 is not a quasi-regular value, consider the blowup $\big(\tilde{U}, \tilde{\omega}, \tilde{\Phi}\big)$. As noted in the proof of Theorem 4.8, for a suitable choice of the blowup parameters and a sufficiently small quasi-regular value ν of Φ, $\tilde{U}_\nu$ is symplectomorphic to M_ν and moreover $\tilde{L}_\nu$ is isomorphic to the pullback of L_ν. As 0 is a quasi-regular value of $\tilde{\Phi}$, we have $\mathrm{RR}(M_\nu, L_\nu) = \mathrm{RR}\big(\tilde{U}_0, \tilde{L}_0\big)$, which is by definition equal to $\mathrm{RR}(M_0, L_0)$. □

Proof of Theorem 2.5. This follows from Theorem 2.7 just as in the abelian case. □

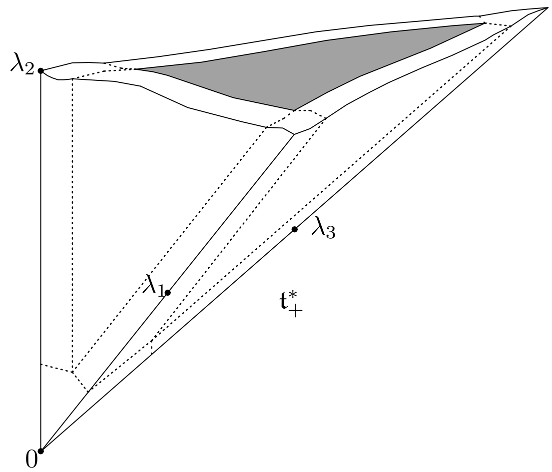

DIAGRAM 3. Admissible subdivision for SU(4), intersected with Weyl chamber. Shaded area represents shifted Weyl chamber

6.2. **Multiplicities.** This section contains the proofs of Theorems 2.9 and 2.14. The main ingredient is the nonabelian gluing formula.

Definition 6.5. An *admissible* or *G-admissible polyhedral subdivision* of $\mathfrak{t}^*_+$ is a collection $\mathfrak{P}$ satisfying the following conditions: every element of $\mathfrak{P}$ is a G-admissible polyhedron in $\mathfrak{t}^*$, their union contains $\mathfrak{t}^*_+$, for every element of $\mathfrak{P}$ all its closed faces are in $\mathfrak{P}$, and the intersection of any two elements of $\mathfrak{P}$ is a closed face of each.

Theorem 6.6 (gluing formula, [34]). *Let $\mathfrak{P}$ be an admissible polyhedral subdivision of $\mathfrak{t}^*_+$. Then*

$$\mathrm{RR}(M, L) = \sum_{\mathcal{P}\in\mathfrak{P}} (-1)^{\operatorname{codim}\mathcal{P}}\,\mathrm{RR}(M_{\mathcal{P}}, L_{\mathcal{P}}) \tag{6.1}$$

as virtual characters of G. ☐

An example of an admissible polyhedral subdivision of $\mathfrak{t}^*_+$ is the subdivision that is dual to the decomposition into walls, which can be described as follows. For $\sigma \preccurlyeq \tau$ define the polyhedral cone $\mathcal{C}_{\sigma\tau}$ in $\mathfrak{t}^*$ to be the product of $\mathfrak{a}^*$ and the cone in $[\mathfrak{g},\mathfrak{g}]^*$ spanned by the vectors

$$-\alpha_{j_1}, -\alpha_{j_2}, \dots, -\alpha_{j_r} \quad \text{and} \quad \lambda_{i_1}, \lambda_{i_2}, \dots, \lambda_{i_s}. \tag{6.2}$$

Here $r = \operatorname{codim}\sigma$, $s = \dim\bigl(\tau \cap [\mathfrak{g},\mathfrak{g}]^*\bigr)$, α_{j_1}, $\alpha_{j_2}, \dots, \alpha_{j_r}$ are the positive simple roots perpendicular to $\sigma \cap [\mathfrak{g},\mathfrak{g}]^*$, and λ_{i_1}, $\lambda_{i_2}, \dots, \lambda_{i_s}$ are the fundamental weights spanning the wall $\tau \cap [\mathfrak{g},\mathfrak{g}]^*$. Hence

$$\operatorname{codim}\mathcal{C}_{\sigma\tau} = \dim\tau - \dim\sigma. \tag{6.3}$$

Now choose λ in the interior of the Weyl chamber, let $\mathcal{P}_{\sigma\tau}$ be the shifted cone $\lambda + \mathcal{C}_{\sigma\tau}$, and let $\mathfrak{P}_\lambda$ be the collection of all $\mathcal{P}_{\sigma\tau}$ (see Diagram 3):

$$\mathfrak{P}_\lambda = \{\, \mathcal{P}_{\sigma\tau} : \sigma \preccurlyeq \tau \preccurlyeq \mathfrak{t}^*_+ \,\}.$$

Lemma 6.7. *For generic λ in* $\operatorname{int}\mathfrak{t}^*_+$, $\mathfrak{P}_\lambda$ *is an admissible polyhedral subdivision of* $\mathfrak{t}^*_+$.

Outline of proof. The closed faces of $\mathcal{P}_{\sigma\tau}$ are the $\mathcal{P}_{\rho\upsilon}$ with $\rho \preccurlyeq \sigma$ and $\tau \preccurlyeq \upsilon$, so $\mathfrak{P}_\lambda$ is closed under inclusion of faces.

The intersection of $\mathcal{P}_{\sigma\tau}$ and $\mathcal{P}_{\rho\upsilon}$ is

$$\mathcal{P}_{\sigma\tau} \cap \mathcal{P}_{\rho\upsilon} = \mathcal{P}_{\sigma\wedge\rho,\tau\vee\upsilon},$$

where $\sigma \wedge \rho$ is the largest open face contained in $\bar{\sigma} \cap \bar{\rho}$ and $\tau \vee \upsilon$ is the smallest open face that contains τ and υ in its closure. This implies that $\mathfrak{P}_\lambda$ is closed under taking intersections.

Every wall σ is contained in the union of all $\mathcal{P}_{\upsilon\tau}$ with $\upsilon \preccurlyeq \tau \preccurlyeq \sigma$, so $\mathfrak{P}_\lambda$ covers $\mathfrak{t}^*_+$. In fact,

$$\sigma \cap \mathcal{P}_{\upsilon\tau} \neq \emptyset \quad \text{if and only if} \quad \upsilon \preccurlyeq \tau \preccurlyeq \sigma. \tag{6.4}$$

It follows from (6.2) that the tangent space to $\mathcal{P}_{\upsilon\tau}$ contains the annihilator of $\mathfrak{a}_\upsilon$. Therefore, if $\upsilon \preccurlyeq \tau \preccurlyeq \sigma$, then the tangent space to $\mathcal{P}_{\upsilon\tau}$ contains $\mathfrak{a}^0_\sigma$. We conclude from (6.4) that all polyhedra in $\mathfrak{P}_\lambda$ satisfy condition 1 of Definition 6.4 for all M. Furthermore, $\upsilon \preccurlyeq \tau$ implies that the sets $\{i_1, i_2, \dots, i_r\}$ and $\{j_1, j_2, \dots, j_s\}$ are disjoint, so the vectors (6.2) are linearly independent, and consequently any (minimal) set of labels defining $\mathcal{P}_{\upsilon\tau}$ has constant excess. Because condition 2 of Definition 6.4 is satisfied generically, we conclude that $\mathfrak{P}_\lambda$ is an admissible polyhedral subdivision of $\mathfrak{t}^*_+$ for generic values of λ. $\square$

From (6.4) we obtain $\mathfrak{t}^*_+ \cap \mathcal{P}_{\sigma\tau} \subset \operatorname{star}\tau$ whenever $\sigma \preccurlyeq \tau$. This means that for all $\tau \neq \mathfrak{a}^*$ the symplectic cut $M_{\mathcal{P}_{\sigma\tau}}$ possesses a nontrivial global cross-section, namely the symplectic cut $Y_{\sigma\tau} = (Y_\tau)_{\mathcal{P}_{\sigma\tau}}$ of Y_τ. Let $L_{\sigma\tau} = (L|_{Y_\sigma})_{\mathcal{P}_{\sigma\tau}}$ denote the corresponding cut bundle. From Proposition 6.3, the gluing formula and (6.3) we obtain

$$\mathrm{RR}(M, L) = \sum_{\substack{\sigma,\tau \\ \sigma\preccurlyeq\tau\preccurlyeq\mathfrak{t}^*_+}} (-1)^{\dim\tau - \dim\sigma} \operatorname{Ind}^G_{G_\tau} \mathrm{RR}(Y_{\sigma\tau}, L_{\sigma\tau}). \tag{6.5}$$

Henceforth let υ denote the principal wall for M as defined in Section 3.2, so that $\Delta \subset \bar{\upsilon}$. The *principal cone* in $\mathfrak{P}_\lambda$ is $\mathcal{P}_{\upsilon\upsilon}$ and the *principal cut* of M is $M_{\mathcal{P}_{\upsilon\upsilon}}$. The global cross-section $Y_{\upsilon\upsilon}$ of $M_{\mathcal{P}_{\upsilon\upsilon}}$ is a Hamiltonian G_υ-orbifold. Recall that $[G_\upsilon, G_\upsilon]$ acts trivially on $Y_{\upsilon\upsilon}$, so that $\mathrm{RR}(Y_{\upsilon\upsilon})$ is a trivial $[G_\upsilon, G_\upsilon]$-character and hence

$$\operatorname{Ind}^G_{G_\upsilon} \mathrm{RR}(Y_{\upsilon\upsilon}, L_{\upsilon\upsilon}) = \operatorname{Ind}^G_T \mathrm{RR}(Y_{\upsilon\upsilon}, L_{\upsilon\upsilon}).$$

Note further that all terms in (6.5) vanish except those for which $\sigma \preccurlyeq \tau \preccurlyeq \upsilon$. For certain line orbibundles the pair $(Y_{\upsilon\upsilon}, L_{\upsilon\upsilon})$ captures all the information needed to compute $\mathrm{RR}(M, L)$.

Theorem 6.8 (abelianization)**.** *Suppose that L has the property that for all $m \in M$ the action of the identity component of $G_m \cap \left[G_{\Phi(m)}, G_{\Phi(m)}\right]$ on the fibre L_m is trivial. Let υ be the principal wall for M. If λ is generic and sufficiently small, then*

$$\mathrm{RR}(M, L) = \operatorname{Ind}^G_T \mathrm{RR}(Y_{\upsilon\upsilon}, L_{\upsilon\upsilon}).$$

According to Lemma 3.11 the assumption on L is satisfied for rigid, moment and dual moment bundles.

Proof. Let $\tau \preccurlyeq \upsilon$. We assert that if λ is sufficiently small, then for all $\sigma \preccurlyeq \tau$ the G_τ-equivariant orbibundle $L_{\sigma\tau}$ on the G_τ-orbifold $Y_{\sigma\tau}$ is rigid with respect to the subgroup $[G_\tau, G_\tau]$. Note that $Y_{\sigma\tau}$ can be regarded as the symplectic cut of $Y_{\tau\tau}$ with respect to $\mathcal{P}_{\sigma\tau}$, so it is enough to show this for $\sigma = \tau$. The symplectic cross-section theorem enables us to reduce this case to the case $\tau = \mathfrak{a}^*$, where we need to show that for λ small L induces a $[G,G]$-rigid orbibundle on $Y_{\tau\tau} = \Phi_+^{-1}(\lambda + \mathfrak{a}^*)/T \cap [G,G]$. Now notice that the condition imposed on L implies that L is almost equivariantly locally trivial on $\Phi^{-1}(\mathfrak{a}^*)$ with respect to the action of $[G,G]$. Therefore, by Lemma 3.7, L is $[G,G]$-almost equivariantly locally trivial on $\Phi^{-1}(G\lambda + \mathfrak{a}^*)$ for sufficiently small λ. The $[G,G]$-rigidity of $L_{\tau\tau}$ is now proved in the same way as in 3 of Lemma 5.10.

It follows from the rigidity of $L_{\sigma\tau}$ and part 1 of Theorem 2.7 that $\mathrm{RR}(Y_{\sigma\tau}, L_{\sigma\tau})$ is constant as a $[G_\tau, G_\tau]$-character and can therefore be regarded as a character of A_τ or T. Since $Y_{\sigma\tau}$ is equal to the symplectic cut of $(Y_{\tau\tau})_{\mathcal{P}_{\sigma\tau}}$, over the points in $\mathcal{P}_{\sigma\tau} \cap \Delta$ it has the same G_τ-symplectic quotients as $Y_{\tau\tau}$. A fortiori, it has the same $[G_\tau, G_\tau]$-symplectic quotients and therefore by the equivariant version of 2 of Theorem 2.7 (where we take $G = [G_\tau, G_\tau]$ and $H = A_\tau$) we have $\mathrm{RR}(Y_{\sigma\tau}, L_{\sigma\tau}) = \mathrm{RR}(Y_{\tau\tau}, L_{\tau\tau})$ as characters of A_τ, and hence as characters of T, for all $\sigma \preccurlyeq \tau$ such that $\mathcal{P}_{\sigma\tau} \cap \Delta$ is nonempty. From (6.5) we now conclude

$$\mathrm{RR}(M, L) = \sum_{\tau \preccurlyeq \upsilon} \sum_{\substack{\sigma \preccurlyeq \tau \\ \mathcal{P}_{\sigma\tau} \cap \Delta \neq \emptyset}} (-1)^{\dim \tau - \dim \sigma} \operatorname{Ind}_T^G \mathrm{RR}(Y_{\tau\tau}, L_{\tau\tau}).$$

The result now follows from the combinatorial identities

$$\sum_{\substack{\sigma \preccurlyeq \tau \\ \mathcal{P}_{\sigma\tau} \cap \Delta \neq \emptyset}} (-1)^{\dim \tau - \dim \sigma} = \begin{cases} 1 & \text{if } \tau = \upsilon, \\ 0 & \text{otherwise,} \end{cases}$$

which derive from the fact that for any simplicial cone $\mathcal{C}$ the sum $\sum_{\mathcal{F} \preccurlyeq \mathcal{C}} (-1)^{\dim \mathcal{F}}$ is equal to 1 if $\mathcal{C}$ is a point and 0 otherwise. □

Proof of Theorem 2.9. Choose a generic λ in the principal wall υ. We can choose λ so small that whenever $0 < t \leq 1$ the subdivision $\mathfrak{P}_{t\lambda}$ is G-admissible and Theorem 6.8 holds with λ replaced by $t\lambda$. Let us denote by Y^t the global cross-section of the principal cut M with respect to $\mathfrak{P}_{t\lambda}$ and by L^t the corresponding orbibundle. In view of Theorem 6.8 and the fact that $\chi_\mu = \operatorname{Ind}_T^G \zeta_\mu$ we need only show that

$$\mathrm{RR}(Y^t, L^t) = \sum_{\mu \in \Lambda^* \cap \Delta} \mathrm{RR}\big(M_\mu, L_\mu^{\mathrm{shift}}\big)\, \zeta_\mu. \tag{6.6}$$

Now L^t is not a moment bundle on Y^t, so the abelian result proved in Section 5.2.2 does not directly apply. Notice however that $\mathfrak{L} = \bigcup_{0<t\leq 1} L^t$ is an *asymptotic* moment bundle on $\mathfrak{Y} = \bigcup_{0<t\leq 1} Y^t$ as defined in Section 5.2.3. Using the notation of that section we obtain from Theorem 5.11 that for $0 < t \leq 1$

$$\mathrm{RR}(Y^t, L^t) = \sum_{\mu \in \Lambda^* \cap \Delta} \mathrm{RR}\big(Y^t_{\gamma_\mu(t)}, (L^t)^{\mathrm{shift}}_{\gamma_\mu(t)}\big)\, \zeta_\mu.$$

In addition, it follows from the cross-section theorem that the quotients $Y^t_{\gamma_\mu(t)}$ and $M_{\gamma_\mu(t)}$ are isomorphic and also $(L^t)^{\mathrm{shift}}_{\gamma_\mu(t)} \cong L^{\mathrm{shift}}_{\gamma_\mu(t)}$, so

$$\mathrm{RR}\big(Y^t_{\gamma_\mu(t)}, (L^t)^{\mathrm{shift}}_{\gamma_\mu(t)}\big) = \mathrm{RR}\big(M_{\gamma_\mu(t)}, L^{\mathrm{shift}}_{\gamma_\mu(t)}\big) = \mathrm{RR}\big(M_\mu, L^{\mathrm{shift}}_\mu\big)$$

by Theorem 2.5. This proves (6.6). ☐

Proof of Theorem 2.14. The proof of the multiplicity formula (2.8) is completely analogous to the proof of Theorem 2.9. The formula implies that the support of the multiplicity function is contained in the orbit of $-\operatorname{int}\Delta$ under the affine action $w \odot \mu = w(\mu + \rho) - \rho$ of $\mathfrak{W}$,

$$\operatorname{supp} N_{L^{-1}} \subset -\rho + \bigcup_{w \in \mathfrak{W}} w(\rho - \operatorname{int}\Delta). \tag{6.7}$$

Since $\operatorname{int}\Delta$ is entirely contained in the principal open wall σ of M, it follows from Lemma 6.9 below that the intersection of the right-hand side of (6.7) with Λ^*_+ is contained in $*\big(\operatorname{int}\Delta - 2(\rho - \rho_\sigma)\big)$. ☐

We thank Dan Barbasch for helping us prove the following lemma.

Lemma 6.9. *Let λ be a dominant weight and let σ be the open wall of $\mathfrak{t}^*_+$ containing λ. Let w_σ be the longest element in the Weyl group $\mathfrak{W}_\sigma \subset \mathfrak{W}$ of the centralizer G_σ. Then the following conditions are equivalent.*

1. *There exists $w \in \mathfrak{W}$ such that $w \odot (-\lambda)$ is dominant*;
2. *$\lambda - \rho$ is regular*;
3. *$w_\sigma(\lambda - \rho)$ is dominant regular*;
4. *$\lambda - 2(\rho - \rho_\sigma)$ is dominant.*

If 1 holds, then $w = w_0 w_\sigma$ and $w \odot (-\lambda) = \big(\lambda - 2(\rho - \rho_\sigma)\big)^* = \lambda^* - 2(\rho - \rho^*_\sigma)$.

Proof. Let R be the root system of G and $R_\sigma = \{\alpha \in R : (\lambda, \check\alpha) = 0\}$ the root system of G_σ. Let R^+ and R^+_σ denote the corresponding sets of positive roots. Note that w_σ fixes σ, so $w_\sigma\lambda = \lambda$. Furthermore, w_σ permutes the elements of $R^+ - R^+_\sigma$ and sends R^+_σ to R^-_σ. This implies

$$w_\sigma\rho = \rho - 2\rho_\sigma \qquad \text{and} \qquad w_\sigma(\lambda - \rho) = \lambda + 2\rho_\sigma - \rho. \tag{6.8}$$

If $w \odot (-\lambda)$ is dominant for some $w \in \mathfrak{W}$, then $w(-\lambda + \rho)$ is dominant regular, so $\lambda - \rho$ is regular. This shows that 1 implies 2. The implications $3 \Rightarrow 4 \Rightarrow 1$ are obvious from (6.8).

Next we show that 2 implies 3. It suffices to show that $(\lambda - \rho, \check\alpha) \geq 0$ for all w_σ-positive roots α. If $\alpha = -\beta$ with $\beta \in R^+_\sigma$, then $(\lambda - \rho, \check\alpha) = (\rho, \check\beta) > 0$. If $\alpha \in R^+ - R^+_\sigma$, then can write $\alpha = \beta_1 + \beta_2 + \cdots + \beta_k$, where $\beta_1, \beta_2, \ldots, \beta_k$ are simple, β_1 is not in R^+_σ, and every partial sum $\alpha_i = \beta_1 + \beta_2 + \cdots + \beta_i$ is in R^+. Note that $(\lambda, \check\alpha_1)$ is a positive integer and $(\lambda - \rho, \check\alpha_1) = (\lambda, \check\alpha_1) - 1$, so $(\lambda - \rho, \check\alpha_1) \geq 0$. Note also that $(\lambda - \rho, \check\alpha_{i+1} - \check\alpha_i)$ is either positive or equal to -1 for every i. This implies that if $(\lambda - \rho, \check\alpha) = (\lambda - \rho, \check\alpha_k)$ was negative, then $(\lambda - \rho, \check\alpha_i)$ would be equal to 0 for some i, which contradicts the regularity of $\lambda - \rho$. We conclude that $(\lambda - \rho, \check\alpha) \geq 0$.

Finally, if 1 holds, then $-w(\lambda - \rho)$ is dominant regular, and so is $w_\sigma(\lambda - \rho)$ by 3. It follows that $w = w_0 w_\sigma$ and hence $w \odot (-\lambda) = \lambda^* - 2\rho + 2\rho^*_\sigma$ by (6.8). ☐

Appendix A. Normal forms

Section A.1 contains a brief review of minimal coupling and some observations on deformation equivalence and equivariant blowing up. In Section A.2 we prove a relative version of the constant-rank embedding theorem and a number of related embedding and deformation results.

A.1. **Minimal coupling.** Let (B, ω_B) be a symplectic orbifold and $P \xleftarrow{\pi} B$ a principal H-orbibundle, where H is a compact Lie group. Let $\theta \in \Omega^1(P, \mathfrak{h})$ be a principal connection on P and (Q, ω_Q) a Hamiltonian H-orbifold with moment map $\Phi_Q \colon Q \to \mathfrak{h}^*$. Let $\mathrm{pr}_{P,Q}$ denote the projection from $P \times Q$ onto P, resp. Q, and let $\langle\cdot,\cdot\rangle \colon \mathfrak{h}^* \times \mathfrak{h} \to \mathbb{R}$ be the dual pairing. Then the two-form

$$\mathrm{pr}_P^* \pi^* \omega_B + \mathrm{pr}_Q^* \omega_Q + \mathrm{d}\langle \mathrm{pr}_Q^* \Phi_Q, \mathrm{pr}_P^* \theta\rangle$$

on the principal orbibundle

$$H \longrightarrow P \times Q \longrightarrow P \times^H Q$$

is basic, and therefore descends to a closed two-form on the associated orbibundle $P \times^H Q$, which is called the *minimal coupling form*. The following result is due to Sternberg. A proof for the manifold case can be found in [20]. The generalization to orbifolds is straightforward.

Theorem A.1. 1. *The minimal coupling form is nondegenerate in a neighbourhood of $P \times^H \Phi_Q^{-1}(0)$.*
2. *A Hamiltonian G-action on B with moment map $\Psi_B \colon B \to \mathfrak{g}^*$ that lifts to an action on P by θ-preserving principal orbibundle automorphisms induces a Hamiltonian G-action on $P \times^H Q$. The moment map on $P \times^H Q$ is a sum $\Phi_B + \Phi_\theta$, where Φ_B and Φ_θ are defined as follows: $\langle \Phi_B, \xi\rangle$ and $\langle \Phi_\theta, \xi\rangle$ are the functions induced by the H-invariant functions $\mathrm{pr}_P^* \pi^* \langle \Psi_B, \xi\rangle$, resp. $\langle \mathrm{pr}_Q^* \Phi_Q, \mathrm{pr}_P^* \iota(\xi_P)\theta\rangle$.*
3. *A Hamiltonian action of a Lie group G' on Q with moment map $\Psi_Q \colon Q \to (\mathfrak{g}')^*$ that commutes with the action of H induces an G'-action on $P \times^H Q$, which is Hamiltonian with moment map induced by the H-invariant map $\mathrm{pr}_Q^* \Psi_Q$.*

Here an *automorphism* of a fibre orbibundle refers to a diffeomorphism of the total space that preserves the structure group and maps fibres to fibres and hence induces a diffeomorphism from the base onto itself. (In the case of the principal orbibundle P this simply means an H-equivariant diffeomorphism of the total space.)

Weinstein observed that the associated orbibundle can also be obtained as a symplectic quotient. The *universal phase space* of the principal orbibundle P is the orbifold $P \times \mathfrak{h}^*$. It carries a closed two-form $\omega_\theta = \mathrm{pr}_P^* \pi^* \omega_B + \mathrm{d}\langle \mathrm{pr}_{\mathfrak{h}^*}, \mathrm{pr}_P^* \theta\rangle$, which is nondegenerate in a neighbourhood of $P \times \{0\}$. The H-action on the universal phase space is Hamiltonian with moment map given by $\mathrm{pr}_{\mathfrak{h}^*}$. The H-action on $P \times \mathfrak{h}^* \times Q$ is therefore Hamiltonian with moment map given by

$$\Psi(p, \beta, q) = \beta + \Phi_Q(q).$$

Since the H-action on P is locally free, the symplectic quotient $(P \times \mathfrak{h}^* \times Q)/\!\!/H$ is a symplectic orbifold. The map $(p, q) \mapsto (p, -\Phi_Q(q), q)$ is an H-equivariant diffeomorphism onto $\Psi^{-1}(0)$, and therefore descends to a diffeomorphism

$$P \times^H Q \longrightarrow (P \times \mathfrak{h}^* \times Q)/\!\!/H,$$

which one can easily show to be symplectic (with respect to the minimal coupling form defined by the connection on P). Sometimes the form ω_θ on the universal phase space is globally nondegenerate.

Example A.2. Suppose H is a subgroup of G acting on G by right multiplication. Let B be the symplectic manifold $T^*(G/H)$ and define P to be the pullback of the bundle $B \to G/H$ under the projection $G \to G/H$. Let us identify T^*G with $G \times \mathfrak{g}^*$ by means of left-invariant one-forms. Then $B \cong G \times^H \mathfrak{h}^0$, where $\mathfrak{h}^0$ is the annihilator of $\mathfrak{h}$ in $\mathfrak{g}^*$, so $P \cong G \times \mathfrak{h}^0$ is a principal H-bundle over B. Choose a G-equivariant connection θ on P, that is an H-equivariant splitting $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Then $\mathfrak{h}^0 \cong \mathfrak{m}^*$ and we can identify $\mathfrak{h}^*$ with a subspace of $\mathfrak{g}^*$. We thus obtain a diffeomorphism from $P \times \mathfrak{h}^* \cong G \times \mathfrak{h}^0 \times \mathfrak{h}^*$ to $T^*G \cong G \times \mathfrak{g}^*$, and it is easy to see that the pullback of the standard symplectic form on T^*G is equal to ω_θ. Thus ω_θ is symplectic globally. We conclude that the associated bundle

$$F(H,Q) = G \times^H (\mathfrak{h}^0 \times Q) \cong P \times^H Q \cong (T^*G \times Q)/\!\!/H \tag{A.1}$$

is a symplectic orbifold. (In this example $F(H,Q)$ is not merely an orbibundle but a genuine locally trivial fibre bundle, whose fibres happen to be orbifolds.)

Now note that the action of G on itself by left multiplication lifts to a Hamiltonian G-action on $B = T^*(G/H)$ and to an action on P by orbibundle automorphisms that preserve the connection. By 2 of Theorem A.1 we obtain a Hamiltonian G-action on $F(H,Q)$ with moment map given by

$$[g,\beta,q] \longmapsto g(\beta + \Phi_Q(q)), \tag{A.2}$$

where $[g,\beta,q]$ denotes the H-orbit through $(g,\beta,q) \in G \times \mathfrak{h}^0 \times Q$. The zero level set is therefore the bundle $G \times^H \Phi_Q^{-1}(0)$, and $F(H,Q)/\!\!/G \cong Q/\!\!/H$ (reduction in stages).

Finally, let K be a compact Lie subgroup of $\mathrm{Diff}(Q)$ containing the image of H under the action map $\rho\colon H \to \mathrm{Diff}(Q)$. Suppose that K acts on Q in a Hamiltonian fashion and that the H-moment map Φ_Q is equal to the K-moment map followed by the natural map $\rho^*\colon \mathfrak{k}^* \to \mathfrak{h}^*$. Let $N_{G\times K}(H)$ be the normalizer of H under the embedding $H \to G \times K$ given by $h \mapsto (h, \rho(h))$. Then the quotient

$$K(H,Q) = N_{G\times K}(H)/H \tag{A.3}$$

acts on $F(H,Q)$ in a Hamiltonian fashion, and the action commutes with the action of G. This group has a particularly simple interpretation.

Lemma A.3. *The group $K(H,Q)$ is canonically isomorphic to $\mathrm{Aut}\big(F(H,Q)\big)^G$, the group of G-equivariant automorphisms of the fibre bundle $F(H,Q)$ that preserve the structure group K.* □

Example A.4. As in Section 4.1.1 let S be a locally closed G-invariant symplectic suborbifold of the Hamiltonian G-orbifold M with normal bundle N and let $\mathcal{K}$ be a G-invariant compact subset of M such that $\mathcal{K} \cap S$ is closed. Choose a G-invariant complex structure on N; then $N \cong P \times^{\mathrm{U}(k)} \mathbb{C}^k$, where P is the unitary frame orbibundle of N. According to 2 of Theorem A.1, a neighbourhood of the zero section in N is a Hamiltonian G-orbifold. Its moment map $\Phi_N = \Phi_S + \Phi_\theta$ has the property that $\Phi_N^{-1}(0) \cap S = \Phi_S^{-1}(0)$.

As noted in Section 4.1.1 there exist $\delta > 0$ and a G-invariant open neighbourhood U of $\mathcal{K}$ such that over $U \cap S$ the minimal coupling form is nondegenerate on the disc bundle $N(\delta)$ and $N(\delta)|_{U\cap S}$ embeds properly, equivariantly and symplectically into U. For all $\varepsilon < \delta$ the blowup $\mathrm{Bl}(U,S,\omega,j,\theta,\iota,\varepsilon)$ of U along S is well-defined. Let us now assume that the projection pr_S preserves the level set $\Phi_N^{-1}(0)$, in other

words, $\mathrm{pr}_S \Phi_N^{-1}(0) = \Phi_S^{-1}(0)$, or equivalently, $\Phi_N^{-1}(0)$ is equal to $\Phi_S^{-1}(0) \cap \Phi_\theta^{-1}(0)$. This implies that $\Phi_N^{-1}(0)$ is a conical subset of N, because Φ_B is constant along the fibres of pr_S and Φ_θ is quadratic. If $\varepsilon_0 < \varepsilon_1$ we can retract the orbifold $U - N(\varepsilon_0)$ smoothly and equivariantly onto $U - N(\varepsilon_1)$ by pushing points outward along the fibres of N. This retraction leaves the complement of $N(\delta)$ in U fixed and preserves the zero level set of Φ_N. It therefore induces a deformation equivalence between the ε_0-blowup and the ε_1-blowup.

Lemma A.5. *If the projection* pr_S *preserves the zero fibre of* Φ_N, *then for all* ε_0 *and* $\varepsilon_1 < \delta$ *the blowups* $\mathrm{Bl}(U, S, \omega, j, \theta, \iota, \varepsilon_0)$ *and* $\mathrm{Bl}(U, S, \omega, j, \theta, \iota, \varepsilon_1)$ *are deformation equivalent as Hamiltonian* G*-orbifolds.* □

A.2. **Embedding theorems.** In this section we present an addendum to the constant-rank embedding theorem and some other embedding and deformation results that rely on a straightforward extension of Moser's method for proving the Darboux Theorem. Moser's method shows that under certain conditions the fact that two symplectic forms are deformation equivalent implies that they are strongly isotopic. It is a trivial observation that the isotopy obtained by Moser's method "depends smoothly on parameters". This leads to a relative version of the constant-rank embedding theorem and also enables us in certain cases to deform a path of diffeomorphisms to a path of symplectomorphisms.

The details are as follows. Let $\mathfrak{B}$ be an orbifold and let $\pi\colon \mathfrak{M} \to \mathfrak{B}$ be a fibre orbibundle over $\mathfrak{B}$. A *relative symplectic form* on $\mathfrak{M}$ or a *symplectic form on* $\mathfrak{M}$ *over* $\mathfrak{B}$ is a two-form ω on the vertical tangent orbibundle $T_{\mathrm{vert}}\mathfrak{M} = \ker \mathrm{d}\pi$, the restriction of which to every fibre of π is closed and nondegenerate. It is obvious how to define, in the presence of G-actions on $\mathfrak{B}$ and $\mathfrak{M}$ such that π is equivariant, a *relative moment map* $\Phi\colon \mathfrak{M} \to \mathfrak{g}^*$ for the action.

Consider a smooth path ω_t of relative symplectic forms on $\mathfrak{M}$ defined for $0 \leq t \leq 1$. Denote by $\dot\omega_t$ the t-derivative of ω_t and suppose that $\dot\omega_t = \mathrm{d}_\pi \sigma_t$, where $\sigma\colon [0,1] \to \Gamma\big(\Lambda^1 T_{\mathrm{vert}}\mathfrak{M}\big)$ is a smooth path of vertical one-forms on $\mathfrak{M}$, and d_π denotes the vertical exterior derivative. Define a time-dependent vertical vector field Ξ_t on M by

$$\Xi_t = -\omega_t^\flat \sigma_t,$$

where $\omega_t^\flat\colon (T_{\mathrm{vert}}\mathfrak{M})^* \to T_{\mathrm{vert}}\mathfrak{M}$ denotes the lowering operator associated to ω, and let ψ_t be its flow. Then clearly ψ_t preserves the fibres of π and $\mathcal{L}(\Xi_t)\omega_t = -\dot\omega_t$, so

$$\psi_t^* \omega_t = \omega_0 \tag{A.4}$$

for those t and those points of $\mathfrak{M}$ at which the flow ψ_t is defined.

Sometimes there is a natural choice for the one-forms σ_t and an a priori estimate for the existence interval of the flow ψ_t.

Example A.6. Let $\mathfrak{Y}$ be a fibre orbibundle over $\mathfrak{B}$ and $\mathfrak{Y} \to \mathfrak{M}$ a locally closed embedding that commutes with the projections $\mathfrak{Y} \to \mathfrak{B}$ and $\mathfrak{M} \to \mathfrak{B}$. Regard $\mathfrak{Y}$ as a subset of $\mathfrak{M}$ and suppose that $\dot\omega_t|_{\mathfrak{Y}} = 0$. Let $\mathfrak{O} \to \mathfrak{B}$ be a relative tubular neighbourhood of $\mathfrak{Y}$ (i. e. the fibres of $\mathfrak{O}$ are tubular neighbourhoods of the fibres of $\mathfrak{Y}$) and define a homotopy $I\colon \mathfrak{O} \times [0,1] \to \mathfrak{O}$ by radial retraction along the fibres of the orbibundle projection $\mathrm{pr}_{\mathfrak{Y}}\colon \mathfrak{O} \to \mathfrak{Y}$. Then $I_1 = \mathrm{id}_{\mathfrak{O}}$ and $I_0 = \mathrm{pr}_{\mathfrak{Y}}$. Let κ_I be the associated chain homotopy on the de Rham complex of $\mathfrak{O}$, which is given by $\kappa_I \alpha = \int \iota(\partial/\partial t) I^* \alpha \, \mathrm{d}t$ for $\alpha \in \Omega(\mathfrak{O})$. Put $\sigma_t = \kappa_I \dot\omega_t$. Then

$$\mathrm{d}_\pi \sigma_t = \mathrm{d}_\pi \kappa_I \dot\omega_t = \mathrm{d}_\pi \kappa_I \dot\omega_t + \kappa_I \mathrm{d}_\pi \dot\omega_t = \dot\omega_t - \mathrm{pr}_{\mathfrak{Y}}^*\big(\dot\omega_t|_{\mathfrak{Y}}\big) = \dot\omega_t.$$

Furthermore, $\Xi_t = 0$ on $\mathfrak{Y}$, so $\mathfrak{Y}$ is fixed under the flow ψ_t. It follows that there exists an open neighbourhood $\mathfrak{O}'$ of $\mathfrak{Y}$ contained in $\mathfrak{O}$ such that the flow ψ_t is defined for all $t \in [0,1]$ and all initial values in $\mathfrak{O}'$.

As a first application of the relative Moser method, we show how in special circumstances an isotopy can be deformed to a symplectic isotopy.

Example A.7. In the setting of Example A.6, let $\mathfrak{B} = [0,1]$, $\mathfrak{M} = M \times \mathfrak{B}$, and $\mathfrak{Y} = Y \times \mathfrak{B}$. Here M is an orbifold and Y a locally closed suborbifold of M. Let ω be a fixed symplectic form on M and let $F\colon O \times \mathfrak{B} \to O$ be an isotopy of a tubular neighbourhood O of Y leaving Y pointwise fixed. Assume that F is symplectic at all points of Y in the sense that $(F_b^*\omega)_y = \omega_y$ for all $b \in \mathfrak{B}$ and all y in Y. (We do not assume that F starts at the identity.) Put $\mathfrak{O} = O \times \mathfrak{B}$. Define a path of vertical two-forms ω_t on $\mathfrak{O}$ by putting $\omega_t = (1-t)\omega + tF_b^*\omega$ on $O \times \{b\} \subset \mathfrak{O}$. Then $(\omega_t)_y = \omega_y$ for all $y \in \mathfrak{Y}$, so ω_t is symplectic on a neighbourhood of $\mathfrak{Y}$ in $\mathfrak{O}$. Furthermore, on $Y \times \{b\}$ we have $\dot\omega_t = (-\omega + F_b^*\omega) = 0$, so $\dot\omega_t = 0$ on $\mathfrak{Y}$. We are therefore in the situation of Example A.6 and obtain a flow $\psi_t\colon \mathfrak{O}' \to \mathfrak{O}$ which is defined for all t and for all initial values in a small open $\mathfrak{O}'$ containing $\mathfrak{Y}$ and satisfies (A.4). We may assume $\mathfrak{O}'$ is of the form $O' \times \mathfrak{B}$ for some open subset O' of O containing Y. For $b \in \mathfrak{B}$ let $\rho_b\colon O' \to O' \times \{b\}$ be the diffeomorphism $\rho_b(m) = (m,b)$. For each $t \in [0,1]$ define an isotopy $F^{(t)}\colon O' \times \mathfrak{B} \to O$ by $F_b^{(t)} = F_b\rho_b^{-1}\psi_t\rho_b$. Then $F^{(0)} = F$ and $\bigl(F_b^{(1)}\bigr)^*\omega = \bigl(\rho_b^{-1}\bigr)^*\psi_1^*\rho_b^*F_b^*\omega = \omega$, so $F^{(1)}$ is symplectic. This means that we have constructed a path of isotopies joining $F = F^{(0)}$ to the symplectic isotopy $F^{(1)}$.

We note two additional properties of ψ_t. Firstly, if F_b is a symplectomorphism for some $b \in \mathfrak{B}$, then on $O' \times \{b\}$ we have $\sigma_t = \kappa_I\dot\omega_t = 0$ for all t and so $\Xi_t = 0$. Therefore the flow ψ_t is trivial on $O' \times \{b\}$. Secondly, suppose that ω is invariant under the action of a compact Lie group K which leaves Y invariant and preserves the projection $O \to Y$. Then if F_b is K-equivariant for some $b \in \mathfrak{B}$, the flow ψ_t is equivariant on $O' \times \{b\}$.

In this context we also have the following elementary result.

Lemma A.8. *Every symplectic isotopy $O' \times [0,1] \to O$ that leaves Y fixed and starts at the identity is Hamiltonian.*

Proof. Let F be such an isotopy. Consider its infinitesimal generator η_b (where $b \in [0,1]$) and note that the one-form $\beta = \iota(\eta_b)\omega$ is closed because F is symplectic, and that $\beta|_Y = 0$ because Y is fixed under H. The function $\kappa_I\beta$ satisfies $\mathrm{d}\kappa_I\beta = \mathrm{d}\kappa_I\beta + \kappa_I\mathrm{d}\beta = \beta - \mathrm{pr}_Y^*(\beta|_Y) = \beta = \iota(\eta_b)\omega$. In other words, the vector field η_b is generated by the time-dependent Hamiltonian $\kappa_I\beta$. ∎

The following lemma is used in Section 4.1.1. We use the notation of that section and of Example A.4.

Lemma A.9. *For $\delta' < \delta$ let $f\colon N(\delta') \to N(\delta)$ be a symplectic map restricting to the identity on S. Then there exist $\delta'' < \delta'$ and a symplectic isotopy $H\colon N(\delta'') \times [0,1] \to N(\delta)$ such that $H_0 = f$, H_1 is S^1-equivariant, $H_b|_S = \mathrm{id}_S$ for all $b \in [0,1]$. If f is G-equivariant, then H can be chosen to be equivariant.*

Proof. Below we construct an isotopy $F\colon N(\delta') \times [0,1] \to U$ fixing S such that $F_0 = f$, F_1 is S^1-equivariant and $F_b^*\omega_m = \omega_m$ for all $m \in S$ and $0 \le b \le 1$. Then we put $H = F^{(1)}$ as in Example A.7 above. As we have seen, H is a symplectic

isotopy, and since $F_0 = f$ preserves the symplectic form, the flow ψ_t is trivial on $N(\delta') \times \{b\}$ for all t, so $H_0 = F_0 \circ \mathrm{id} = f$. Furthermore, since F_1 is equivariant, so is H_1.

The construction of the isotopy F is in two stages. Between time $b = 0$ and $1/2$ we isotop f to its fibre derivative $T_N f$ by means of the obvious isotopy $F(m, b) = (1-2b)^{-1} f\big((1-2b)m\big)$. The fact that f is a symplectic map and leaves S fixed implies that for all $m \in S$ the derivative $T_m f$ preserves the direct sum decomposition of (uniformized) tangent spaces $\tilde{T}_m N = \tilde{T}_m S \oplus \tilde{T}_0 N_m$. It follows from this that f has the same derivative as $T_N f$ at all points in the zero section S, and in fact $T_m F_b = T_m f$ for all $b \in [0, 1/2]$ and all m in S. Consequently, $(F_b^* \omega)_m = (f^* \omega)_m = \omega_m$ for all b and m.

The second half of the isotopy comes about as follows. Note that $T_N f$ is an element of $\mathrm{Aut}_S(N)$, that is the group of linear automorphisms of N that preserve the symplectic forms on the fibres and restrict to the identity on S. If P is the Hermitian frame orbibundle of N, then $\mathrm{Aut}_S(N)$ can be viewed as the space of sections of the associated orbibundle $P \times^{\mathrm{U}(n)} \mathrm{Sp}(2n, \mathbb{R})$. Using the retraction of $\mathrm{Sp}(2n, \mathbb{R})$ onto its maximal compact subgroup $\mathrm{U}(n)$ we can construct a path in $\mathrm{Aut}_S(N)$ defined for $1/2 \leq b \leq 1$ starting at $T_N f$ and ending at a Hermitian automorphism of N. Observe that symplectic orbibundle automorphisms preserve the symplectic form at all points of the zero section and that Hermitian automorphisms commute with the scalar S^1-action. By composing the two isotopies we obtain the requisite isotopy F.

It is not hard to check that each step in this proof can be made equivariant with respect to the action of G. It follows that the isotopy H can be made equivariant. □

Another application of the relative Moser method is the relative Darboux-Moser-Weinstein Theorem: if ω_0 and ω_1 are relative symplectic forms on a fibre orbibundle $\pi\colon \mathfrak{M} \to \mathfrak{B}$ such that $\omega_{0,y} = \omega_{1,y}$ for all y in a locally closed suborbibundle $\mathfrak{Y}$ of $\mathfrak{M}$, then there exist open neighbourhoods $\mathfrak{U}_0$ and $\mathfrak{U}_1$ of $\mathfrak{Y}$ and a diffeomorphism $f\colon \mathfrak{U}_0 \to \mathfrak{U}_1$ commuting with π such that $f(y) = y$, $\mathrm{d}_\pi f_y = \mathrm{id}_{T_y \mathfrak{M}}$ for all $y \in \mathfrak{Y}$, and $f^* \omega_1 = \omega_0$. The proof is word for word the same as in the absolute case, relying on linear interpolation between ω_0 and ω_1. In turn this leads to relative versions of all the usual embedding theorems in symplectic geometry.

As an example we state the relative constant-rank embedding theorem. Let $\mathfrak{Z}$ be a fibre orbibundle over $\mathfrak{B}$ and let τ be a vertical two-form on $\mathfrak{Z}$ that is closed on every fibre. Assume that τ has constant rank on $\mathfrak{Z}$. Assume further that G acts on $\mathfrak{B}$, that $\mathfrak{Z}$ is an equivariant orbibundle, and that the action on $\mathfrak{Z}$ is Hamiltonian in the sense that there exists a G-equivariant map $\Phi_{\mathfrak{Z}}\colon \mathfrak{Z} \to \mathfrak{g}^*$ satisfying $\langle \mathrm{d}_\pi \Phi_{\mathfrak{Z}}, \xi \rangle = \iota(\xi_{\mathfrak{Z}})\tau$. Let $\mathfrak{N}$ be a G-equivariant symplectic vector orbibundle over $\mathfrak{Z}$ with fibre symplectic form σ. Now let ω be a relative symplectic form on $\mathfrak{M}$ and assume G acts on $\mathfrak{M}$ in a Hamiltonian fashion with relative moment map Φ. An *embedding of* $\mathfrak{Z}$ *into* $\mathfrak{M}$ *with normal bundle* $\mathfrak{N}$ is an embedding of fibre orbibundles $\iota\colon \mathfrak{Z} \to \mathfrak{M}$ such that $\iota^* \omega = \tau$, $\iota^* \Phi = \Phi_{\mathfrak{Z}}$, and the pullback under ι of the relative symplectic normal bundle of $\iota(\mathfrak{Z})$ in $(\mathfrak{M}, \omega)$ is isomorphic to $(\mathfrak{N}, \sigma)$.

The *standard* embedding $\mathfrak{Z} \hookrightarrow \mathfrak{Y}$ with normal bundle $\mathfrak{N}$ is constructed as follows. As an orbifold, $\mathfrak{Y}$ is the total space of the direct sum $\mathfrak{S} \oplus \mathfrak{N}$, where $\mathfrak{S}$ is the orbibundle on $\mathfrak{Z}$ dual to the suborbibundle $\ker \tau$ of the vertical tangent bundle $T_{\mathrm{vert}} \mathfrak{Z}$.

The relative symplectic form and moment map $(\omega_{\mathfrak{Y}}, \Phi_{\mathfrak{Y}})$ on $\mathfrak{Y}$ are constructed in two stages.

At the first stage one chooses a section s of the orbibundle map $T^*_{\mathrm{vert}}\mathfrak{Z} \to \mathfrak{S}$ and defines a closed two-form $\omega_{\mathfrak{S}}$ on the fibres of the projection $\mathfrak{S} \to \mathfrak{B}$ by $\omega_{\mathfrak{S}} = \mathrm{pr}_Z\, \tau + s^*\Omega$, where Ω is the standard symplectic form on the fibres of $T^*_{\mathrm{vert}}\mathfrak{Z} \to \mathfrak{B}$. Near $\mathfrak{Z}$ the form $\omega_{\mathfrak{S}}$ is nondegenerate in the vertical direction, and the G-action on $\mathfrak{S}$ is Hamiltonian with moment map given by $\Phi_{\mathfrak{S}} = \mathrm{pr}^*_{\mathfrak{Z}}\, \Phi_{\mathfrak{Z}} + s^*\Phi_{\mathrm{vert}}$, where $\langle \Phi_{\mathrm{vert}}(p), \xi \rangle = p(\xi_{\mathfrak{Z}})$, the standard moment map on $T^*_{\mathrm{vert}}\mathfrak{Z}$.

At the second stage one notices that as an orbifold $\mathfrak{Y} = \mathfrak{S} \oplus \mathfrak{N}$ is identical to the total space of the pullback of $\mathfrak{N}$ along the map $\mathfrak{S} \to \mathfrak{Z}$, and by means of (fibrewise) minimal coupling constructs a relative symplectic form $\omega_{\mathfrak{Y}}$ on $\mathfrak{Y}$, using the relative symplectic form $\omega_{\mathfrak{S}}$ on the base $\mathfrak{S}$, an invariant σ-compatible almost complex structure $\mathfrak{J}$ on $\mathfrak{N}$, and a (relative) connection one-form θ on the orbibundle $\mathfrak{P}$ of $\mathfrak{J}$-unitary frames on $\mathfrak{N}$. By Theorem A.1 the G-action on $\mathfrak{Y}$ is Hamiltonian with respect to $\omega_{\mathfrak{Y}}$ with relative moment map $\Phi_{\mathfrak{Y}}$, and it is straightforward to check that the zero section $\mathfrak{Z} \hookrightarrow \mathfrak{Y}$ is an embedding of $\mathfrak{Z}$ with normal bundle $\mathfrak{N}$. The relative version of the constant-rank embedding theorem is now proved in the same way as the absolute version; cf. e. g. [41].

Theorem A.10 (relative constant-rank embeddings)**.** *For every embedding ι of $\mathfrak{Z}$ into $\mathfrak{M}$ with normal bundle $\mathfrak{N}$ there exist a G-invariant open neighbourhood $\mathfrak{U}$ of $\mathfrak{Z}$ in $\mathfrak{Y}$ and an isomorphism of relative Hamiltonian G-orbifolds*

$$f\colon (\mathfrak{U}, \omega_{\mathfrak{Y}}, \Phi_{\mathfrak{Y}}) \longrightarrow (\mathfrak{M}, \omega, \Phi)$$

onto an open neighbourhood of $\iota(\mathfrak{Z})$ in $\mathfrak{M}$ such that the diagram

$$\begin{tikzcd} \mathfrak{Z} \arrow[d] \arrow[dr, "\iota"] & \\ \mathfrak{U} \arrow[r, "f"'] & \mathfrak{M} \end{tikzcd}$$

commutes. ∎

The zero section $\mathfrak{Z} \hookrightarrow \mathfrak{S}$ is a coisotropic embedding of $\mathfrak{Z}$ and the zero section $\mathfrak{S} \hookrightarrow \mathfrak{Y}$ is a symplectic embedding of $\mathfrak{S}$. It is not hard to see that $\mathfrak{S}$ is a *minimal* symplectic suborbifold of $\mathfrak{Y}$ containing $\mathfrak{Z}$ in the sense that if $\mathfrak{S}'$ is a locally closed symplectic suborbifold of $\mathfrak{Y}$ such that $\mathfrak{Z} \subset \mathfrak{S}' \subset \mathfrak{S}$, then $\mathfrak{S}'$ is open in $\mathfrak{S}$. Theorem A.10 thus proves the existence of minimal symplectic suborbifolds containing a given constant-rank suborbifold. To what extent are minimal symplectic suborbifolds unique? We shall answer this question in the absolute case only, though even there the proof uses the relative constant-rank embedding theorem.

Theorem A.11. *Let (M, ω, Φ) be a Hamiltonian G-orbifold and let Z be a G-invariant compact suborbifold of constant rank. Let S_0 and S_1 be minimal G-invariant locally closed symplectic suborbifolds of M containing Z. Then there exist an automorphism $f\colon M \to M$ of the Hamiltonian G-orbifold M and G-invariant open neighbourhoods U_0 and U_1 of Z such that f fixes Z and maps $U_0 \cap S_0$ onto $U_1 \cap S_1$.*

Proof. Let N be the symplectic normal bundle of Z in M, let $Z \hookrightarrow Y$ be the standard embedding of Z with normal bundle N, and choose an embedding $U \hookrightarrow M$ as in Theorem A.10. Below we find an equivariant symplectic isotopy $F\colon U' \times$

$[0,1] \to U$ of an invariant open U' such that $Z \subset U' \subset U$, F_0 is the identity on U', and F leaves Z fixed. According to Lemma A.8, F is generated by a time-dependent Hamiltonian vector field. We extend this vector field to M by multiplying its Hamiltonian function by a smooth cutoff function that is supported on U' and identically equal to 1 on a smaller $U'' \subset U'$. Since Z is compact, the resulting Hamiltonian vector field is compactly supported and hence integrates to a globally defined isotopy $\check{F}$ of M; and $f = \check{F}_1$ is the desired automorphism.

To construct the isotopy F, we may without loss of generality replace M with the model space Y and assume S_0 to be the suborbibundle $S = (\ker \tau)^*$ of Y. This means that we can identify S_0 with the orbibundle $TS_0|_Z$ over Z. The construction is in two steps. First we find a symplectic isotopy of U that fixes Z and moves $S_0 = TS_0|_Z$ to $TS_1|_Z$, and then we construct F in the special case where $TS_0|_Z = TS_1|_Z$.

Step 1. Regard Y as an orbibundle over Z and note that both $TS_0|_Z$ and $TS_1|_Z$ are suborbibundles of Y that are complementary to N. We can therefore select a path of suborbibundles $N_t^\perp$ of Y defined for $0 \le t \le 1$ such that $N_0^\perp = TS_0|_Z$, $N_1^\perp = TS_1|_Z$, and $N_t^\perp$ is complementary to N for all t. We assert that, for all t, near the zero section the total space of $N_t^\perp$ is a symplectic suborbifold of Y and that Z is coisotropic in $N_t^\perp$. This is proved by showing that $TN_t^\perp|_Z$ is a symplectic suborbibundle of $TY|_Z$, as follows. By construction, $TY|_Z$ is canonically a symplectic direct sum $R \oplus K \oplus K^* \oplus N$, where $K = \ker \tau$, $R = TZ/K$, and the orbibundle $K \oplus K^*$ carries the canonical symplectic form on its fibres. The fact that $TN_t^\perp|_Z$ is symplectic now follows from the first assertion of Lemma A.12. We then apply the relative *coisotropic* embedding theorem, that is to say, we apply Theorem A.10 with $\mathfrak{B} = [0,1]$, $\mathfrak{Z} = Z \times \mathfrak{B}$, $\mathfrak{N} = 0$, $\mathfrak{Y} = S_0 \times \mathfrak{B}$, and $\mathfrak{M} = \bigcup_t N_t^\perp \times \{t\}$. As a result we obtain a G-invariant open neighbourhood $\mathfrak{O}$ of $\mathfrak{Z}$ in $\mathfrak{Y}$ and an isomorphism of relative Hamiltonian G-orbifolds

$$h \colon (\mathfrak{O}, \omega_{\mathfrak{Y}}, \Phi_{\mathfrak{Y}}) \longrightarrow (\mathfrak{M}, \omega, \Phi)$$

onto an open neighbourhood of $\mathfrak{Z}$ in $\mathfrak{M}$ such that the relevant commutative diagram commutes. We can choose $\mathfrak{O}$ to be of the form $O \times [0,1]$, where $Z \subset O \subset S_0$. In other words, h is (the track of) a symplectic isotopy of O which fixes Z and h_t maps S_0 to $N_t^\perp$. After composing h_t with the map $h_0^{-1} \colon S_0 \to S_0$ we may also assume that h starts at the identity. We can view h as an embedding of $\mathfrak{O}$ into $Y \times [0,1]$ and as such want to extend it to an isotopy of a full neighbourhood U of Z in Y. This is achieved by applying the relative *symplectic* embedding theorem to the embedding of the relative symplectic manifold $\mathfrak{O}$ into $Y \times [0,1]$. To this end we need to calculate the symplectic normal bundle $\mathfrak{E}$ of $S_0 \times [0,1]$ in $Y \times [0,1]$. Let π denote the projection $S_0 \to Z$. The restriction of $\mathfrak{E}$ to $S_0 \times \{0\}$ is equal to $\pi^* N$, which is by definition equal to Y, considered as a symplectic orbibundle over S_0. The unit interval being contractible, we conclude that $\mathfrak{E}$ is isomorphic to $Y \times [0,1]$, considered as a symplectic orbibundle over $S_0 \times [0,1]$. By the relative symplectic embedding theorem, h lifts to a symplectic embedding H of $U' \times [0,1]$ into $Y \times [0,1]$, where $U' \subset U$ is an open subset of Y containing Z, as in the following commutative

diagram:

$$\begin{tikzcd}
\mathrm{pr}_Z^* N \arrow[d] & Y \times [0,1] \arrow[l, hook'] \arrow[d] \arrow[dr, dotted, tail, "H"] & \\
Z \times [0,1] & S_0 \times [0,1] \arrow[l, hook', "\pi\times\mathrm{id}"'] \arrow[r, tail, "h"] & Y \times [0,1].
\end{tikzcd}$$

By composing H with the projection $Y \times [0,1] \to Y$ we find the desired isotopy moving $TS_0|_Z$ to $TS_1|_Z$.

Step 2. We may henceforth assume that $S_0 = TS_0|_Z = TS_1|_Z$. Let O be an open subset of S_0 containing Z and let $h\colon O \to S_1$ be any diffeomorphism onto an open subset of S_1 that fixes Z and satisfies $T_x h = \mathrm{id}$ for all x in Z. Such a map can be found for instance by choosing a projection map p of $TY|_Z$ onto $TS_0|_Z$; the restriction of p to S_1 has derivative equal to the identity at all points of Z and can therefore be locally inverted. Let $H(x,b) = b^{-1}h(bx)$ be the isotopy deforming h to its fibre derivative; then $H_0 = \mathrm{id}$, $H_1 = h$, and $T_x H_b = \mathrm{id}$ for all x in Z. Consider the forms $(1-t)\omega + tH_b^*\omega$ on S_0. Applying Moser's trick with parameter b as in Example A.6 we find an open $O' \subset O$ and an isotopy $I\colon O' \times [0,1] \to O$ such that $I|_Z = \mathrm{id}$, $I_0 = \mathrm{id}$, and $I_b^* H_b^* \omega = \omega$. The isotopy $\check{H}\colon O' \times [0,1] \to Y$ defined by $\check{H}_b = H_b I_b$ therefore satisfies $\check{H}_0 = \mathrm{id}$, $\check{H}_1 = hI_1$ maps $O' \subset S_0$ to S_1 and $\check{H}_b^*\omega = \omega$. This symplectic isotopy can now be extended to a neighbourhood U' of Z in Y by use of the relative symplectic embedding theorem, as in Step 1 above. $\square$

Lemma A.12. *Let K be a vector space and let R and N symplectic vector spaces. Let V be the symplectic direct sum $R \oplus K \oplus K^* \oplus N$, where $K \oplus K^*$ carries the canonical symplectic form. Let $N^\perp$ be any complementary subspace to N in $K^* \oplus N$. Then $R \oplus K \oplus N^\perp$ is a symplectic subspace of V. Let $P\colon V \to N$ be the linear projection with kernel $R \oplus K \oplus N^\perp$. Then the restriction of P to $(R \oplus K \oplus N^\perp)^\omega$ is a symplectic isomorphism onto N.*

Proof. It clearly suffices to prove this for $R = 0$. Let $d = \dim K$, $2n = \dim V$. There exists a symplectic basis e_1, $e_2, \ldots,$ e_n, f_1, $f_2, \ldots,$ f_n of V such that e_1, $e_2, \ldots,$ e_d form a basis of K, f_1, $f_2, \ldots,$ f_d form a basis of K^*, and e_{d+1}, $e_{d+2}, \ldots,$ e_n, f_{d+1}, $f_{d+2}, \ldots,$ f_n form a basis of N. Furthermore, if $a_i = Pf_i$ for $i = 1, 2, \ldots,$ d, then $f_1 - a_1$, $f_2 - a_2, \ldots,$ $f_d - a_d$ form a basis of $N^\perp$. Put $f_i' = f_i - a_i + \sum_{j=1}^d \alpha_{ij} e_j$ for $i = 1, 2, \ldots,$ d, where $\alpha_{ij} = \frac{1}{2}\omega(a_i, a_j)$. It is easy to check that e_1, $e_2, \ldots,$ e_d, f_1', $f_2', \ldots,$ f_d' form a symplectic basis of $K \oplus N^\perp$, so $K \oplus N^\perp$ is symplectic.

It is also easy to see that the vectors

$$e_i' = e_i + \sum_{j=1}^d \omega(e_i, a_j) e_j,$$

$$f_i' = e_i + \sum_{j=1}^d \omega(f_i, a_j) e_j,$$

defined for $i = d+1, d+2, \ldots, n$, are a symplectic basis of $(K \oplus N^\perp)^\omega$. Clearly, the projection P sends e_i' to e_i and f_i' to f_i and therefore maps $(K \oplus N^\perp)^\omega$ symplectically onto N. $\square$

Appendix B. A product formula

The following result was proved in a holomorphic context by Borel in Appendix II of [21].

Theorem B.1. *Let B be a compact almost complex orbifold and let X be an almost complex fibre orbibundle over B with compact general fibre Y and orbibundle projection π. Assume that the structure group of X can be reduced to a compact Lie group. Then*

$$\mathrm{RR}(X, \pi^* E) = \mathrm{RR}(B, E)\, \mathrm{RR}(Y, \mathbb{C})$$

for every complex vector orbibundle E over B.

We establish a slightly stronger result, namely an integration formula, Theorem B.3, for the Todd form of the vertical tangent bundle. For simplicity we present the proof in the manifold category; the proof for orbifolds is analogous.

B.1. **Cartan map and equivariant curvature.** Let K be a compact (but not necessarily connected) Lie group, let $\mathbb{C}[\mathfrak{k}]$ be the graded algebra of polynomials on $\mathfrak{k}$, and let Y be a K-manifold. We denote by

$$\Omega_K^k(Y) = \bigoplus_{i+2j=k} \big(\Omega^i(Y) \otimes \mathbb{C}[\mathfrak{k}]_j\big)^K$$

the $\mathbb{Z}$-graded algebra of equivariant differential forms. We also consider the $\mathbb{Z}_2$-graded algebra

$$\hat{\Omega}_K(Y) = \big(\Omega(Y) \otimes \mathbb{C}[[\mathfrak{k}]]\big)^K$$

of equivariant forms with coefficients in the formal power series $\mathbb{C}[[\mathfrak{k}]]$. These algebras carry a differential of degree 1 defined by

$$(\mathrm{d}_K \alpha)(\xi) = \mathrm{d}\alpha(\xi) - \iota(\xi_Y)\alpha(\xi).$$

The cocycles in $\Omega_K^*(Y)$ and $\hat{\Omega}_K(Y)$ are denoted by $\mathcal{Z}_K^*$ and $\hat{\mathcal{Z}}_K$, respectively, and the cohomology groups by H_K^* and $\hat{H}_K$, respectively.

Now let $P \to B$ be a K-principal bundle with connection $\theta \in \Omega^1(P, \mathfrak{k})^K$. The curvature of θ is the basic two-form $F^\theta = \mathrm{d}\theta + \frac{1}{2}[\theta, \theta] \in \Omega^2(P, \mathfrak{k})^K$. It can be viewed as a K-equivariant map $\mathfrak{k}^* \to \Omega^2(P)$ and as such extends uniquely to an equivariant multiplicative map $\mathbb{C}[[\mathfrak{k}]] \to \Omega^*(P)$. In other words, given $\alpha \in \hat{\Omega}_K(P)$ we can substitute the curvature in the $\mathfrak{k}$-slot to get a K-invariant differential form $\alpha(F^\theta)$ and thus we obtain a map $j^\theta \colon \hat{\Omega}_K(P) \to \Omega(P)^K$. Let $\mathrm{hor}^\theta \colon \Omega^*(P) \to \Omega^*_{\mathrm{hor}}(P)$ be the projection onto the horizontal forms defined by the connection θ. The composition

$$\mathrm{Car}^\theta = \mathrm{hor}^\theta \circ j^\theta \colon \hat{\Omega}_K(P) \longrightarrow \Omega^*_{\mathrm{basic}}(P) \cong \Omega^*(B)$$

is called the *Cartan map*. Neither hor^θ nor j^θ is a cochain map, but Car^θ is. Its restriction to $\mathbb{C}[[\mathfrak{k}]] \subset \hat{\Omega}_K(P)$ is known as the *Chern-Weil map*.

Consider the product $P \times Y$ and the associated bundle

$$X = P \times^K Y.$$

Let $\mathrm{pr}_P \colon P \times Y \to P$ be the projection onto the first factor and $\mathrm{Car}^{\mathrm{pr}_P^* \theta} \colon \hat{\Omega}_K(P \times Y) \to \Omega^*(X)$ the Cartan map for the principal K-bundle $P \times Y \to X$. If Y is

compact, we can integrate forms over the fibres and thus obtain a diagram

$$\begin{tikzcd} \hat{\Omega}_K(Y) \arrow[r, "\mathrm{pr}_Y^*"] \arrow[d, "\int_Y"'] & \hat{\Omega}_K(P \times Y) \arrow[r, "\mathrm{Car}^{\mathrm{pr}_P^* \theta}"] \arrow[d, "\int_Y"'] & \Omega^*(X) \arrow[d, "\int_Y"'] \\ \hat{\Omega}_K(\mathrm{pt}) \arrow[r, "\mathrm{pr}_{\mathrm{pt}}^*"] & \hat{\Omega}_K(P) \arrow[r, "\mathrm{Car}^\theta"] & \Omega^*(B). \end{tikzcd} \tag{B.1}$$

We assert that this diagram is commutative. The commutativity of the square on the left is obvious; for the commutativity of the square on the right it suffices to show that

$$\int_Y \circ \operatorname{hor}^{\mathrm{pr}_P^* \theta} = \operatorname{hor}^\theta \circ \int_Y \qquad \text{and} \qquad \int_Y \circ j^{\mathrm{pr}_P^* \theta} = j^\theta \circ \int_Y .$$

These identities follow from the fact that the forms $\mathrm{pr}_P^* \theta \in \Omega^1(P \times Y, \mathfrak{k})^K$ and $F^{\mathrm{pr}_P^* \theta} = \mathrm{pr}_P^* F^\theta \in \Omega^2(P \times Y, \mathfrak{k})^K$ have no components in the Y-direction.

Now let H be another compact Lie group and let Q be an H-principal bundle over Y. Assume that the K-action on Y lifts to an action on Q that commutes with the H-action. Choose a K-invariant connection $\phi \in \Omega^1(Q, \mathfrak{h})^{H \times K}$ on Q. Its *K-equivariant curvature* is the H-basic K-equivariant form $F_K^\phi \in \Omega_K^2(Q, \mathfrak{h})$ defined by

$$F_K^\phi = \mathrm{d}_K \phi + \frac{1}{2}[\phi, \phi] = F^\phi - \Psi^\phi .$$

Here $\Psi^\phi \colon Q \to \mathfrak{k}^*$ is the map defined by $\langle \Psi^\phi, \eta \rangle = \iota(\eta_Q)\phi$ for all $\eta \in \mathfrak{k}$, which, being H-invariant, descends to Y. The connection θ on the K-bundle P and the connection ϕ on the H-bundle Q can be combined to a connection ϕ^θ on the H-bundle $P \times^K Q \to P \times^K Y = X$, and the curvature of ϕ^θ can be expressed in terms of the equivariant curvature of ϕ in the following manner.

Lemma B.2. 1. *The K-horizontal part of $\mathrm{pr}_Q^* \phi$, which is given by*

$$\operatorname{hor}^\theta(\mathrm{pr}_Q^* \phi) = \mathrm{pr}_Q^* \phi - \langle \mathrm{pr}_Q^* \Psi^\phi, \mathrm{pr}_P^* \theta \rangle \in \Omega^1(P \times Q, \mathfrak{k})^K ,$$

is a K-basic H-connection one-form on $P \times Q$ and represents a connection one-form $\phi^\theta \in \Omega^1(P \times^K Q, \mathfrak{h})^H$.

2. *The curvature form $F^{\phi^\theta} \in \Omega^2(P \times^K Q, \mathfrak{h})^H$ of ϕ^θ, regarded as a K-basic $\mathfrak{h}$-valued form on $P \times Q$, is equal to $\mathrm{Car}^{\mathrm{pr}_P^* \theta}(\mathrm{pr}_Q^* F_K^\phi)$, where F_K^ϕ is the K-equivariant curvature of ϕ.*

Proof of 2. By 1 the pullback of F^{ϕ^θ} to $P \times Q$ is equal to the curvature of the connection $\operatorname{hor}^\theta(\mathrm{pr}_Q^* \phi)$, which is equal to the K-basic form

$$F^{\operatorname{hor}^\theta \phi} = F^\phi + \frac{1}{2}\big[\langle \Psi^\phi, \theta \rangle, \langle \Psi^\phi, \theta \rangle\big] - \langle \mathrm{d}\Psi^\phi, \theta \rangle - \langle \Psi^\phi, \mathrm{d}\theta \rangle - \big[\langle \Psi^\phi, \theta \rangle, \phi\big], \tag{B.2}$$

where we are suppressing the pullback maps from the notation. On the other hand

$$j^\theta F^\phi = F^\phi - \langle \Psi^\phi, \mathrm{d}\theta \rangle - \frac{1}{2}\big\langle \Psi^\phi, [\theta, \theta] \big\rangle . \tag{B.3}$$

It is clear that the forms (B.2) and (B.3) agree on K-horizontal vectors, so that (B.2) is the K-horizontal part of (B.3). $\square$

The H-Cartan map $\mathrm{Car}^\phi\colon \hat\Omega_H(Q) \longrightarrow \Omega^*(Y)$ for Q has a K-equivariant analogue

$$\mathrm{Car}^\phi_K\colon \hat\Omega_{H\times K}(Q) \longrightarrow \hat\Omega_K(Y),$$

which is defined by $\mathrm{Car}^\phi_K = \mathrm{hor}^\phi \circ j^\phi_K$, where $j^\phi_K(\alpha) = \alpha(F^\phi_K)$. (Notice that $\alpha(F^\phi_K)$ is well-defined as a formal power series on $\mathfrak{k}$ with values in $\Omega^*(Y)$.) It is a cochain map and therefore induces a map $\hat H_{H\times K}(Q) \to \hat H_K(Y)$. Its restriction to the subalgebra $\mathbb{C}[[\mathfrak{h}]]^H \subset \hat\Omega_{H\times K}(Q)$ is the *K-equivariant Chern-Weil map.*

B.2. **Integration formula for the Todd form.** Let Y be an almost complex K-manifold and choose a K-invariant Hermitian inner product and connection on the tangent bundle of Y. These choices give rise to a unitary frame bundle Q of Y and a principal connection θ on it. Let $H = \mathrm{U}(n)$ and consider the Todd series $\mathrm{Td} \in \mathbb{C}[[\mathfrak{h}]]^H$, which is defined by

$$\mathrm{Td}(x_1, x_2, \dots, x_n) = \prod_{j=1}^n \frac{x_j}{1 - \exp(-x_j)}$$

for $(x_1, x_2, \dots, x_n)$ in $i\mathbb{R}^n \cong \mathfrak{t}$, the Cartan subalgebra of H. The form $\mathrm{Td}(Y) = \mathrm{Car}^\phi(\mathrm{Td}) \in \mathcal{Z}^*(Y)$ is the Todd form of Y.

Let $P \to B$ be any K-principal bundle with connection ϕ and consider the vertical tangent bundle $V = P \times^K TY$ of $X = P \times^K Y$ over B. Then V is the $\mathbb{C}^n$-bundle on X associated to the principal H-fibration $P \times^K Q \to X$, on which we have the connection ϕ^θ, and $\mathrm{Td}(V) = \mathrm{Car}^{\phi^\theta}(\mathrm{Td}) \in \mathcal{Z}^*(X)$ is the Todd form of V. As before let $\int_Y\colon \Omega^*(X) \to \Omega^*(B)$ denote integration over the fibres.

Theorem B.3. *The form $\int_Y \mathrm{Td}(V) \in \Omega^*(B)$ is a constant function on B. Its value is equal to $\int_Y \mathrm{Td}(V) = \int_Y \mathrm{Td}(Y)$.*

Proof. First we reduce the general case to the case where K is connected. Let K^0 be the identity component of K and consider the finite covers $\tilde B = P/K^0$ of B, $\tilde X = P \times^{K^0} Y$ of X and $\tilde V = P \times^{K^0} TY$ of V. Then $\tilde X$ is a bundle over $\tilde B$ with fibre Y and $\tilde V$ is the pullback of V under the covering map $\tilde X \to X$. Clearly $\mathrm{Td}(\tilde V)$ is the pullback to $\tilde X$ of $\mathrm{Td}(V) \in \Omega^*(X)$ and $\int_Y \mathrm{Td}(\tilde V)$ is the pullback to $\tilde B$ of $\int_Y \mathrm{Td}(V) \in \Omega^*(B)$, so it is enough to show that $\int_Y \mathrm{Td}(\tilde V) = \int_Y \mathrm{Td}(Y)$. We may therefore assume K to be connected.

Now consider $\mathrm{Td}_K(Y) = \mathrm{Car}^\phi_K(\mathrm{Td}) \in \hat{\mathcal{Z}}_K(Y)$, the *equivariant* Todd form of Y. It follows from 2 of Lemma B.2 that $\mathrm{Td}(V)$ is the image of $\mathrm{Td}_K(Y)$ under the composite map $\mathrm{Car}^{\mathrm{pr}_P^*\theta} \circ \mathrm{pr}_Y^*\colon \hat\Omega_K(Y) \to \Omega(X)$. We conclude from the commutativity of diagram (B.1) that $\int_Y \mathrm{Td}(V)$ is the image of $\int_Y \mathrm{Td}_K(Y)$ under the map $\mathrm{Car}^\theta \circ \mathrm{pr}_{\mathrm{pt}}^*$. By the Berline-Vergne equivariant index theorem [5], $\int_Y \mathrm{Td}_K(Y) \in \hat\Omega_K(\mathrm{pt}) = \mathbb{C}[[\mathfrak{k}]]^K$ is the equivariant arithmetic genus of Y (here we use that K is connected), which by 1 of Theorem 2.7 is constant and equal to $\int_Y \mathrm{Td}(Y)$. Hence

$$\int_Y \mathrm{Td}(V) = \mathrm{Car}^\theta \circ \mathrm{pr}_{\mathrm{pt}}^* \int_Y \mathrm{Td}(Y) = \int_Y \mathrm{Td}(Y)$$

as an element of $\mathcal{Z}^0(B) \cong \mathbb{R}$. ∎

Proof of Theorem B.1. We can write $X = P \times^K Y$, where K is a compact Lie group and P a principal K-bundle over B. By the Hirzebruch-Riemann-Roch theorem

$$\mathrm{RR}(X, \pi^* E) = \int_X \mathrm{Ch}(\pi^* E) \,\mathrm{Td}(X) = \int_X \pi^* \mathrm{Ch}(E) \,\mathrm{Td}(X).$$

Choose a connection on P such that the induced connection on X is invariant under the almost complex structure. Then we can write $TX = V \oplus TB$, where $V = P \times^K TY$ is the vertical tangent bundle of X over B. Hence $\mathrm{Td}(X) = \mathrm{Td}(V)\,\pi^* \mathrm{Td}(B)$ and

$$\mathrm{RR}(X, \pi^* E) = \int_X \pi^* \bigl(\mathrm{Ch}(E) \,\mathrm{Td}(B)\bigr) \,\mathrm{Td}(V) = \int_B \biggl(\mathrm{Ch}(E) \,\mathrm{Td}(B) \int_Y \mathrm{Td}(V)\biggr).$$

The result now follows from Theorem B.3 and Hirzebruch-Riemann-Roch. $\square$

Appendix C. Notation

G; T	compact connected Lie group; maximal torus
$\mathfrak{W}$; Λ	Weyl group; integral lattice in $\mathfrak{t}$
w_0; $*$	longest Weyl group element; involution $\mu \mapsto \mu^* = -w_0\mu$ of $\mathfrak{t}^*$
$w \odot \mu$	affine action $w \odot \mu = w(\mu + \rho) - \rho$ of $\mathfrak{W}$
Λ^*; Λ^*_+	weight lattice $\mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$; monoid of dominant weights
ζ_μ	character of T defined by $\mu \in \Lambda^*$
χ_μ	irreducible character of G with highest weight $\mu \in \Lambda^*_+$
$\mathrm{Rep}\, G$; Ind_H^G	representation ring; induction functor
$\mathfrak{t}^*_+$; σ	positive Weyl chamber in $\mathfrak{t}^*$; open wall of $\mathfrak{t}^*_+$
$\mathrm{star}\,\sigma$; $\mathfrak{S}_\sigma$	open star $\bigcup_{\tau \succcurlyeq \sigma} \tau$ of σ; natural slice in $\mathfrak{g}^*$ at σ
(M, ω, Φ)	Hamiltonian G-orbifold with moment map
ξ_M	vector field on M induced by $\xi \in \mathfrak{g}$
Δ	Kirwan polytope $\Phi(M) \cap \mathfrak{t}^*_+$
$\mathrm{int}\,\Delta$	relative interior of Δ
L	G-equivariant line orbibundle on M
$\mathrm{RR}(M, L)$	equivariant index of M with coefficients in L
$N_L = N$	multiplicity function of L
M_μ; $M_0 = M/\!\!/G$	symplectic quotient of M at μ; resp. 0
L_μ; $L_0 = L/\!\!/G$	quotient orbibundle at μ; resp. 0
L_μ^{shift}	shifted quotient orbibundle at μ
$M_{\geq 0}$; $L_{\geq 0}$	symplectic cut of M w. r. t. circle action; cut bundle
Y_σ; M_σ	cross-section $\Phi^{-1}(\mathfrak{S}_\sigma)$; its saturation GY_σ
$\mathcal{S}$; $\mathcal{P}$; $\mathcal{F}$	set of labels; polyhedron; open face
$D_{\mathcal{S}}$	Delzant space associated to set of labels $\mathcal{S}$
$\mathrm{Iso}(E_1, E_2)$	isomorphisms from fibre bundle $E_1 \to B_1$ to $E_2 \to B_2$
$\mathrm{Aut}(E)$	automorphisms of a fibre bundle $E \to B$
$\mathrm{Aut}_B(E)$	automorphisms of $E \to B$ that map each fibre to itself

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